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Tackling Problems on Affine Space with Locally Nilpotent Derivations on Polynomial Rings

een wetenschappelijke proeve op het gebied
van de Natuurwetenschappen, Wiskunde en Informatica

Proefschrift

ter verkrijging van de graad van doctor
aan de Katholieke Universiteit Nijmegen,
volgens besluit van het College van Decanen
in het openbaar te verdedigen op
woensdag 14 november 2001
des namiddags om 3:30 precies
door

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geboren op 2 april 1971 te Macharen

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Typeset by LaTeX in Times Roman.

ISBN 90-9015143-5

Preface

This thesis would not be the way it is now without the help of many people. First of all, I would like to thank my supervisor Arno van den Essen. His great enthusiasm and many ideas provided a perfect environment for a Ph.D. student. His influence can probably be found on every single page of this thesis. Also Joost Berson and Stefan Maubach, my fellow Ph.D. students supervised by Arno, deserve thanks for the many discussions we had on the topic of polynomial mappings.

I am also indebted to the reading committee: Harm Derksen, Andrzej Nowicki, and Dave Finston, who have donated their time to read the manuscript of this thesis and who have provided valuable comments.

I would also like to thank my office mates, Engelbert Hubbers, for setting a tremendous example, and Mascha Honsbeek, for always being willing to discuss all subjects at hand, being mathematical or not.

I also thank all other colleagues from the mathematical department and the students from Desda for the pleasant working environment and for the many games of bridge and chess we played.

My parents, Ans and Ben van Rossum deserve praise for always being there for me, whatever I needed. My sister Moniek, her boyfriend Robbie, my aunt Ella and my grandmother also were always supportive of me, even if I had not seen them in ages.

Finally, my girlfriend Mariëlle Stoelinga knows better than anyone what writing a thesis means, but she deserves thanks for much more.

Peter van Rossum

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Introduction

Many interesting problems in algebraic geometry can be formulated in an elementary way, yet are very difficult to handle. For example, the Embedding Problem asks if an embedding from \mathbb{C} into \mathbb{C}^n is always rectifiable, *i.e.*, if it is, up to a change of coordinates, always just the standard embedding. For $n = 2$, this has been proven by Abhyankar and Moh in [AM75] and, for $n \geq 4$, by Jelonek in [Jel87] and by Craighero in [Cra86]. For $n = 3$, Shastri constructed an embedding of \mathbb{R} in \mathbb{R}^3 representing the trefoil knot in [Sha92]. Over \mathbb{R} , this is of course not rectifiable, but it is yet unknown whether or not this embedding is rectifiable over \mathbb{C} .

Another example of an easy looking, but notoriously difficult problem is the Cancellation Problem, which was first posed by Zariski in 1942. Given an algebraic variety V over \mathbb{C} with $V \times \mathbb{C} \cong \mathbb{C}^n$, this asks if V must always be isomorphic to \mathbb{C}^{n-1} . This has been proven for $n = 2$ by Rentschler in [Ren68] and for $n = 3$ by Fujita [Fuj79]. For all other n , the question remains open. This thesis proves that the Cancellation Problem has an affirmative answer for a large class of varieties in dimension 4 and gives a candidate counterexample in dimension 5.

As a final example of such a problem, the Jacobian Conjecture must be mentioned. For this, note that a polynomial automorphism $F: \mathbb{C}^n \rightarrow \mathbb{C}^n$ always satisfies $\det JF \in \mathbb{C}^*$, where JF denotes the Jacobian matrix of F . The Jacobian Conjecture asserts the converse. Since its formulation by Keller in [Kel39] several false proofs of this conjecture have been given and some have even been published. However, the conjecture remains open for all $n \geq 2$.

All of the above mentioned problems can be described using locally nilpotent derivations on polynomial rings. For instance, the Cancellation Problem translates as: given a locally nilpotent derivation D on $\mathbb{C}[X_1, \dots, X_n]$, together with an element $s \in \mathbb{C}[X_1, \dots, X_n]$ such that $D(s) = 1$, does it follow that the kernel of D is isomorphic to $\mathbb{C}[X_1, \dots, X_{n-1}]$? This formulation allows for an obvious generalisation: replace the field \mathbb{C} by some other ring R .

This thesis will mostly be concerned with the Cancellation Problem, both over

\mathbb{C} and over other rings, but also the Embedding Problem, Hilbert's 14th Problem, the Linearisation Problem, the Abhyankar-Sathaye Conjecture, and the Jacobian Conjecture will play their part.

Results and overview of this thesis

Chapter 1 gathers the necessary background knowledge from the literature and introduces most of the above mentioned problems. It also explains the correspondence between the geometrical formulation of the Cancellation Problem and the version phrased in terms of locally nilpotent derivations.

Chapter 2 shows how the Cancellation Problem can be seen as a generalisation of the Quillen-Suslin Theorem. The fact that so-called elementary derivations over a field give an affirmative answer to the Cancellation Problem is actually just the Quillen-Suslin Theorem.

The main technique used in this thesis to attack these problems is described in Chapter 3. The basic idea is to reduce questions about derivations, polynomials maps, *etc.*, over arbitrary \mathbb{Q} -algebras to known theorems over fields. An important result of this chapter is that, over a Hermite \mathbb{Q} -domain, local coordinates are coordinates. Another important result is that, in two variables over a \mathbb{Q} -algebra, residual coordinates are coordinates.

The results obtained in Chapter 3 are applied in Chapter 4 to generalise several well-known theorems, such as the Abhyankar-Moh-Suzuki Theorem, to arbitrary \mathbb{Q} -algebras. It also shows that the Cancellation Problem has an affirmative answer for a large class of locally nilpotent derivations in dimension four, including the triangular ones.

Chapter 5 studies embeddings defined over arbitrary \mathbb{Q} -algebras. It turns out that one can associate to an embedding a locally nilpotent derivation and studying this derivation one can gather knowledge about the embedding and vice versa. For embeddings of R into R^2 , this chapter characterises exactly which embeddings are rectifiable. Furthermore, using Shastri's embedding of the trefoil knot into three-space, this chapter describes a candidate counterexample to the Cancellation Problem in dimension five. This same example yields a candidate counterexample to the Linearisation Problem.

Finally, Chapter 6 uses universal rings to study the degree of the inverse of a polynomial automorphism. It gives a bound on the degree of the inverse of a triangular polynomial map in two variables with Jacobian determinant 1 that is much sharper than any bound obtained before.

Conventions

Throughout this thesis all rings will be commutative and have a unit element, unless explicitly noted otherwise. The letters n , m , i , and j will always denote natural numbers.

Inleiding

Veel interessante problemen in de algebraïsche meetkunde kunnen geformuleerd worden op een elementaire manier, maar zijn toch erg moeilijk om aan te pakken. Bijvoorbeeld, het Inbeddings Probleem (Embedding Problem) stelt de vraag of een inbedding van \mathbb{C} in \mathbb{C}^n altijd rectificeerbaar is, d.w.z., of het tot op een coördinaatverandering altijd de standaard inbedding is. Voor $n = 2$ is dit bewezen door Abhyankar en Moh in [AM75] en voor $n \geq 4$ door Jelonek in [Jel87] en door Craighero in [Cra86]. Voor $n = 3$ heeft Shastri een inbedding van \mathbb{R} in \mathbb{R}^3 geconstrueerd die de zgn. klaverbladknoop representeert ([Sha92]). Over \mathbb{R} is deze inbedding natuurlijk niet rectificeerbaar, maar het is nog onbekend of deze inbedding al dan niet rectificeerbaar is over \mathbb{C} .

Een ander voorbeeld van een makkelijk uitziend, maar erg lastig probleem is het Schrap Probleem (Cancellation Problem). Dit probleem is het eerst genoemd door Zariski in 1942. Gegeven een algebraïsche variëteit V over \mathbb{C} waarvoor geldt dat $V \times \mathbb{C} \cong \mathbb{C}^n$, is het de vraag of V altijd isomorf is met \mathbb{C}^{n-1} . Dit is bewezen voor $n = 2$ door Rentschler in [Ren68] en voor $n = 3$ door Fujita in [Fuj79]. Voor grotere n is het probleem nog steeds open. In dit proefschrift wordt bewezen dat het Schrap Probleem een bevestigend antwoord heeft voor een grote klasse van variëteiten in dimensie 4 en er wordt ook een kandidaat tegenvoorbeeld gegeven in dimensie 5.

Een laatste voorbeeld van zo'n probleem is het Jacobi Vermoeden (Jacobian Conjecture). Merk op dat een inverteerbare veeltermafbeelding $F: \mathbb{C}^n \rightarrow \mathbb{C}^n$ altijd voldoet aan $JF \in \mathbb{C}^*$, waarbij JF de Jacobiaan van F is. Het Jacobi Vermoeden is de bewering dat het omgekeerde ook waar is. Vanaf het moment dat Keller dit vermoeden in [Kel39] formuleerde zijn er verscheidene foute bewijzen van dit vermoeden gegeven, waarvan er enkele zelfs gepubliceerd zijn. Het vermoeden is echter nog steeds open voor alle $n \geq 2$.

Al deze problemen kunnen beschreven worden met behulp van lokaal nilpotente derivaties op veeltermringen. Bijvoorbeeld, het Schrap Vermoeden kan als volgt vertaald worden: gegeven een lokaal nilpotente derivatie D op $\mathbb{C}[X_1, \dots, X_n]$

samen met een element $s \in \mathbb{C}[X_1, \dots, X_n]$ waarvoor geldt dat $D(s) = 1$, volgt het dan dat de kern van D isomorf is met $\mathbb{C}[X_1, \dots, X_{n-1}]$? Deze manier van formuleren suggereert ook meteen een generalisatie: vervang het lichaam \mathbb{C} door een andere ring R .

Dit proefschrift houdt zich vooral bezig met het Schrap Probleem, zowel over \mathbb{C} als over andere ringen, maar ook het Inbeddings Probleem, Hilbert's 14^e Probleem, het Linearisatie Probleem, het Abhyankar-Sathaye Vermoeden en het Jacobi Vermoeden zullen een rol spelen.

Resultaten en overzicht van dit proefschrift

In Hoofdstuk 1 wordt een overzicht gegeven van de benodigde achtergrond informatie uit de literatuur en worden de meeste van bovengenoemde problemen geïntroduceerd. Het hoofdstuk legt ook uit wat het verband is tussen de meetkundige formulering van het Schrap Vermoeden en de versie die geformuleerd is in termen van lokaal nilpotente derivaties.

In Hoofdstuk 2 wordt beschreven hoe het Schrap Probleem gezien kan worden als een generalisatie van de Quillen-Suslin Stelling. Het feit dat het Schrap Probleem voor zogenaamde elementaire derivaties over een lichaam een bevestigend antwoord heeft is eigenlijk niets anders dan de Quillen-Suslin Stelling.

De belangrijkste techniek die in dit proefschrift wordt gebruikt om boven genoemde problemen aan te pakken wordt beschreven in Hoofdstuk 3. Het fundamentele idee is om vragen over derivaties, veeltermafbeeldingen, *etc.*, over willekeurige \mathbb{Q} -algebra's te reduceren tot bekende stellingen over lichamen. Een belangrijk resultaat in dit hoofdstuk is dat, over een Hermite \mathbb{Q} -domein, locale coördinaten coördinaten zijn. Een ander belangrijk resultaat is dat residuale coördinaten in twee variabelen over een \mathbb{Q} -algebra coördinaten zijn.

De resultaten van Hoofdstuk 3 worden gebruikt in Hoofdstuk 4 om verscheidene bekende stellingen, zoals de Abhyankar-Moh-Suzuki Stelling, te generaliseren tot willekeurige \mathbb{Q} -algebra's. Het hoofdstuk laat ook zien dat het Schrap Probleem een bevestigend antwoord heeft voor een grote klasse van lokaal nilpotente derivaties in dimensie 4, waaronder de driehoeksvormderivaties.

Hoofdstuk 5 bestudeert inbeddingen over willekeurige \mathbb{Q} -algebra's. Het blijkt dat men aan een inbedding een lokaal nilpotente derivatie kan toevoegen en het bestuderen van deze derivatie geeft inzicht in de inbedding en omgekeerd. Dit hoofdstuk laat ook precies zien welke inbeddingen van R in R^2 rectificeerbaar zijn. Bovendien, door gebruik te maken van Shastri's inbedding van de klaverbladknoop in de \mathbb{R}^3 , wordt er een kandidaat-tegenvoorbeeld tegen het Schrap Probleem in dimensie vijf geconstrueerd. Dit voorbeeld is ook een kandidaat-tegenvoorbeeld

tegen het Linearisatie Probleem.

Tot slot worden in Hoofdstuk 6 universele ringen gebruikt om de graad van de inverse van een inverteerbare veeltermafbeelding te bestuderen. Er wordt een grens gegeven aan de graad van de inverse van een veeltermafbeelding op driehoeksvorm in twee variabelen met Jacobi determinant gelijk aan 1 die veel scherper is dan elke grens dit hiervoor aangetoond is.

Chapter 1

Preliminaries

This chapter collects the required background knowledge. All of it can be found in the literature and most of it is well-known to the experts in the field. For more background information on polynomial automorphisms, see the book [Ess00] by Van den Essen. For additional information on derivations and their kernels, see the account [Now94] by Nowicki.

1.1 Derivations

Let A be a ring. A *derivation* on A is an additive map $D: A \rightarrow A$ satisfying the *Leibniz rule*: $D(a_1a_2) = a_1D(a_2) + a_2D(a_1)$ for all $a_1, a_2 \in A$. If R is another ring and A is an R -algebra via the ring homomorphism $\varphi: R \rightarrow A$, then such a derivation D is called an *R -derivation* if it furthermore satisfies $D(\varphi(r)) = 0$ for all $r \in R$. The collection of all derivations on A is denoted by $\text{Der}(A)$ and the collection of all R -derivations on A is denoted by $\text{Der}_R(A)$.

If A is an R -algebra, say with generating set S , then an R -derivation D on A is completely determined by the images $D(a)$, $a \in S$ of the generators. In particular, an R -derivation on the polynomial ring $R[X_1, \dots, X_n]$ is completely determined by the images $D(X_i)$ of the variables and hence always has the form

$$D = f_1(X_1, \dots, X_n)\partial_{X_1} + \dots + f_n(X_1, \dots, X_n)\partial_{X_n}.$$

Here the f_i are polynomials over R and ∂_{X_i} denotes the usual partial derivative with respect to the variable X_i .

A derivation D on a ring A is called *locally nilpotent* if for every $a \in A$ there is an $n \in \mathbb{N}$ such that $D^n(a) = 0$. If A is an R -algebra with generating set S ,

then D is locally nilpotent if and only if for all $a \in S$ there is an $n \in \mathbb{N}$ such that $D^n(a) = 0$.

On a polynomial ring $R[X_1, \dots, X_n]$ one can easily identify a simple class of locally nilpotent derivations, namely the *triangular* ones. These are the R -derivations on $R[X_1, \dots, X_n]$ of the form

$$f_1(X_2, \dots, X_n)\partial_{X_1} + f_2(X_3, \dots, X_n)\partial_{X_2} + \dots + f_n\partial_{X_n}$$

for certain polynomials $f_i \in R[X_{i+1}, \dots, X_n]$. An R -derivation D on the polynomial ring $R[X_1, \dots, X_n]$ is said to be *triangularizable* if there exists a ring automorphism $\psi \in \text{Aut}_R(R[X_1, \dots, X_n])$ such that $\psi^{-1} \circ D \circ \psi$ is triangular. Of course, triangularizable derivations are locally nilpotent as well.

Locally nilpotent k -derivations D on a k -algebra A , where k is a field of characteristic zero, have a nice geometrical interpretation. This interpretation is explained in Section 1.2; the following definition plays a central role in it.

Definition 1.1.1. Assume that A is a \mathbb{Q} -algebra and let D be a derivation on A . The *exponential map* of D is the map $\exp TD: A \rightarrow A[[T]]$ defined by

$$\exp TD(f) = \sum_{i=0}^{\infty} \frac{1}{i!} D^i(f) T^i.$$

One easily verifies that $\exp TD$ is in fact a ring homomorphism; see, for example, [Ess00], Proposition 1.2.24. Note that, if D is locally nilpotent, then the image of $\exp TD$ is contained in the polynomial ring $A[T]$. In that case, for every element $a \in A$, φ_a denotes the composition of the map $\exp TD: A \rightarrow A[[T]]$ with the substitution homomorphism $A[[T]] \rightarrow A$ sending T to a .

A *slice* of a derivation D on A is an element $s \in A$ such that $D(s) = 1$. Not every derivation has a slice and not even every locally nilpotent derivation has a slice. Just consider, for instance, $Y\partial_X$ on $k[X, Y]$. Every non-zero locally nilpotent derivation D on A does have a *preslice*, though. This is an element $p \in A$ such that $D(p) \neq 0$, but $D^2(p) = 0$.

The kernel of such a derivation D , denoted by A^D , will be of central importance in this thesis. If a derivation over a \mathbb{Q} -algebra has a slice, then its kernel can easily be computed. The following theorem by Wright ([Wri81], Proposition 2.1) and its corollary (see, for instance, [Ess00], Proposition 1.3.21 and Corollary 1.3.23) are important instruments for this computation.

Theorem 1.1.2. Assume that A is a \mathbb{Q} -algebra and let D be a locally nilpotent derivation on A with a slice $s \in A$. Then

1. $A = A^D[s]$;
2. s is algebraically independent over A^D (and therefore A is in fact a polynomial ring in one variable over A^D);
3. viewing A as a polynomial ring in s over A^D , $D = d/ds$. \square

Corollary 1.1.3. Assume that A is a \mathbb{Q} -algebra and let D be a locally nilpotent derivation on A with a slice $s \in A$. Then $A^D = \varphi_{-s}(A)$. In particular, if R is a \mathbb{Q} -algebra, A is a finitely generated R -algebra, say $A = R[a_1, \dots, a_n]$, and D is an R -derivation, then $A^D = R[\varphi_{-s}(a_1), \dots, \varphi_{-s}(a_n)]$. \square

Notation 1.1.4. Let R be a ring. The polynomial ring in n variables over R will be denoted by $R^{[n]}$. Isomorphism of R -algebras will be denoted by \cong_R .

Proposition 1.1.5. Let R be a \mathbb{Q} -algebra and let D be a locally nilpotent R -derivation on $R[X] := R[X_1, \dots, X_n]$ with a slice in $R[X]$. Then the following two statements are equivalent:

1. $R[X]^D \cong_R R^{[n-1]}$;
2. there is a $\psi \in \text{Aut}_R(R[X])$ such that $\psi^{-1} \circ D \circ \psi = \partial_{X_1}$.

Proof $1 \Rightarrow 2$: Suppose that $R[X_2, \dots, X_n] \rightarrow R[X]^D$ is an R -isomorphism. Using Theorem 1.1.2, this can be extended to an R -isomorphism ψ from $R[X_1, \dots, X_n]$ to $R[X]^D[s] = R[X]$ by sending X_1 to s . Then $\psi^{-1} \circ D \circ \psi = \partial_{X_1}$.

$2 \Rightarrow 1$: Suppose that $\psi: R[X] \rightarrow R[X]$ is an R -automorphism such that $\psi^{-1} \circ D \circ \psi = \partial_{X_1}$. Then $D = \psi \circ \partial_{X_1} \circ \psi^{-1}$ and hence $R[X]^D = \psi(R[X]^{\partial_{X_1}}) = \psi(R[X_2, \dots, X_n]) \cong_R R^{[n-1]}$. \square

A very easy example of a locally nilpotent derivation with a slice is ∂_{X_1} on the polynomial ring $\mathbb{C}[X_1, \dots, X_n]$. One might wonder if this is, up to conjugation, the only example. In view of the previous proposition, this question can be formulated as follows.

Problem 1.1.6 (Cancellation Problem). Let D be a locally nilpotent derivation on $\mathbb{C}[X] := \mathbb{C}[X_1, \dots, X_n]$ and assume that D has a slice in $\mathbb{C}[X]$. Is it then true that $\mathbb{C}[X]^D \cong_{\mathbb{C}} \mathbb{C}^{[n-1]}$?

The next section gives a geometrical interpretation of this problem. This interpretation will also explain the name “Cancellation Problem”.

More generally, one can ask the same question for an arbitrary \mathbb{Q} -algebra R instead of \mathbb{C} . This generalisation will be referred to as the Cancellation Problem over R .

Problem 1.1.7 (Cancellation Problem over R). *Let R be a \mathbb{Q} -algebra and let D be a locally nilpotent R -derivation on $R[X] := R[X_1, \dots, X_n]$ with a slice in $R[X]$. Is it then true that $R[X]^D \cong_R R^{[n-1]}$?*

1.2 Geometrical interpretation

Let k be a field of characteristic zero.

It is possible to give a geometrical interpretation of a locally nilpotent k -derivation D on a finitely generated k -domain. Given an algebraic variety V over k , a G_a -action on V is an algebraic action of the additive algebraic group of k on the variety V , i.e., a regular map $\sigma: k \times V \rightarrow V$ satisfying $\sigma(0, x) = x$ and $\sigma(s + t, x) = \sigma(s, \sigma(t, x))$ for all $s, t \in k$ and all $x \in V$. The geometrical interpretation is then given by the following proposition.

Proposition 1.2.1. *There is a one-one correspondence between G_a -actions on an affine algebraic variety V over k and locally nilpotent k -derivations on its coordinate ring A . The G_a -action corresponding to a locally nilpotent k -derivation D on A has $\exp TD: A \rightarrow A[T] = k[T] \otimes_k A$ as its associated map of coordinate rings. Conversely, if $\sigma: k \times V \rightarrow V$ is a G_a -action, with associated map of coordinate rings $\sigma^*: A \rightarrow A[T]$, then the corresponding locally nilpotent k -derivation D on A is given by $D(f) = \partial_T(\sigma^*(f))[T := 0]$. \square*

This proposition can be proven by straightforward calculation. See, for example, Proposition 9.5.2 of [Ess00].

Consider a G_a -action $\sigma: k \times V \rightarrow V$ on an affine algebraic variety V over k with coordinate ring A . The invariant ring of the G_a -action, denoted by A^{G_a} , is the subring $\{f \in A \mid \forall s \in k \forall x \in V [f(\sigma(s, x)) = f(x)]\}$. Here elements of A are seen as functions from V to k .

Proposition 1.2.2. *Let D be a locally nilpotent k -derivation on the coordinate ring A of some affine algebraic variety V over k . Then the kernel of D equals the invariant ring A^{G_a} . \square*

The Cancellation Problem can now of course be phrased in terms of G_a -actions on \mathbb{C}^n . There is, however, an even more direct geometrical formulation.

Consider an affine algebraic variety V over k with coordinate ring A . Assume that $V \times k \cong_k k^n$. In terms of coordinate rings, this means that $A[T] \cong_k k[X] := k[X_1, \dots, X_n]$; say $\psi: A[T] \rightarrow k[X]$ is a k -isomorphism. Now, let D be the derivation $\psi \circ d/dT \circ \psi^{-1}$ on $k[X]$. Then D is locally nilpotent and has a slice, namely $\psi(T)$. Furthermore, $k[X]^D = \psi(A) \cong_k A$.

Conversely, if D is a locally nilpotent derivation on $k[X]$ with a slice, then $k[X]^D$ is the coordinate ring of some affine algebraic variety V over k satisfying $V \times k \cong_k k^n$, by Theorem 1.1.2.

Now, $V \cong_k k^{n-1}$ if and only if $A \cong_k k^{[n-1]}$ if and only if $k[X]^D \cong_k k^{[n-1]}$. So this gives another, equivalent, way to phrase the Cancellation Problem (over \mathbb{C} , but obviously over every field k of characteristic zero).

Problem 1.2.3 (Cancellation Problem). *Let V be an affine algebraic variety over \mathbb{C} . Assume that $V \times \mathbb{C} \cong_{\mathbb{C}} \mathbb{C}^n$. Is it then true that $V \cong_{\mathbb{C}} \mathbb{C}^{n-1}$?*

1.3 Kernel of a derivation

Let k be a field of characteristic zero and let R be a k -algebra. This thesis will mostly be concerned with locally nilpotent derivations on polynomial rings $k[X_1, \dots, X_n]$ and $R[X_1, \dots, X_n]$. This section gathers a few well-known and less well-known facts about the kernel of derivations on these rings.

There are many interesting problems about derivations on polynomial rings and their kernels. The Cancellation Problem mentioned earlier is one; an other one is the following.

Problem 1.3.1 (Finite Generators Problem). *Let D be a derivation on the polynomial ring $k[X] := k[X_1, \dots, X_n]$. Is it then true that $k[X]^D$ is a k -algebra of finite type?*

This problem is closely related to Hilbert's 14th Problem, which is the following.

Problem 1.3.2 (Hilbert's 14th Problem). *Let L be a subfield of the rational function field $k(X_1, \dots, X_n)$ containing k . Is $L \cap k[X_1, \dots, X_n]$ a k -algebra of finite type?*

Nagata and Nowicki have proven in [NN88] that the Finite Generators Problem is true for $n \leq 3$. Based on Nagata's counterexample to Hilbert's 14th Problem, Derksen constructed an example of a derivation on $\mathbb{C}[X_1, \dots, X_{32}]$ whose kernel was not of finite type ([Der93]). Later Deveney and Finston used Robert's counterexample to Hilbert's 14th Problem to create an example of a locally nilpotent

derivation on a polynomial ring in 7 variables whose kernel was not of finite type. The example they give in [DF94] is

$$D := X^3\partial_S + Y^3\partial_T + Z^3\partial_U + (XYZ)^2\partial_V$$

on the polynomial ring $k[S, T, U, V, X, Y, Z]$.

This result was improved upon by Freudenburg, who constructed a locally nilpotent counterexample in 6 variables ([Fre00]). Later, Daigle and Freudenburg adapted this example to make one in 5 variables ([DF99]), namely

$$D := X^3\partial_S + S\partial_T + T\partial_U + X^2\partial_V$$

on the polynomial ring $k[S, T, U, V, X]$.

The only remaining case of the Finite Generators Problem is, therefore, the case that $n = 4$. Partial results have been obtained by Maubach in [Mau00], where he proved that the kernel of a so-called triangular monomial derivation is always of finite type. The full result is, however, still open.

If the derivation is known to be locally nilpotent, then one has even stronger results about its kernel for small n . In two variables, the following theorem was proven by Rentschler in [Ren68].

Theorem 1.3.3. *Let D be a non-zero locally nilpotent derivation on $k[X_1, X_2]$. Then there is a $\psi \in \text{Aut}_k(k[X_1, X_2])$ and an $f(X_2) \in k[X_2]$ such that $D = \psi^{-1} \circ f(X_2)\partial_{X_1} \circ \psi$. \square*

Note that a derivation on $k[X_1, X_2]$ that is of the form $f(X_2)\partial_{X_1}$ has a slice if and only if $f(X_2)$ is in fact a non-zero constant in k . So, a locally nilpotent derivation on $k[X_1, X_2]$ with a slice is, up to a conjugation, equal to one of the form $a\partial_{X_1}$ for some $a \in k^*$. In particular, its kernel is isomorphic to $k^{[1]}$. This solves the Cancellation Problem in two variables over a field of characteristic zero.

Rentschler's result was improved upon Daigle and Freudenburg in [DF98], who studied locally nilpotent derivations over a UFD. Bhatwadekar and Dutta ([BD97]) considered the case of a Noetherian integral domain and Berson, Van den Esen, and Maubach studied locally nilpotent derivations in two variables over an arbitrary \mathbb{Q} -algebra ([BEM01]). The following result appears in the latter paper. For the formulation of this result, recall that the *divergence* of a derivation D on $R[X_1, \dots, X_n]$ is defined by $\text{div}(D) := \sum_{i=1}^n \frac{\partial D(X_i)}{\partial X_i}$. Note that in two variables, a derivation on $R[X_1, X_2]$ has divergence zero if and only if it is of the form $\frac{\partial p}{\partial X_2}\partial_{X_1} - \frac{\partial p}{\partial X_1}\partial_{X_2}$ for some polynomial $p \in R[X_1, X_2]$.

Theorem 1.3.4. *Let R be a \mathbb{Q} -algebra and let D be a non-zero locally nilpotent derivation on $R[X_1, X_2]$ with divergence 0, say $D = \frac{\partial p}{\partial X_2} \partial_{X_1} - \frac{\partial p}{\partial X_1} \partial_{X_2}$ with $p \in R[X_1, X_2]$, and assume that $1 \in (D(X_1), D(X_2))$. Then D has a slice and $R[X_1, X_2]^D = R[p]$. \square*

A locally nilpotent R -derivation D on $R[X_1, X_2]$ for which there exists a slice $s \in R[X_1, X_2]$ always satisfies $1 \in (D(X_1), D(X_2))$, since $1 = \frac{\partial s}{\partial X_1} D(X_1) + \frac{\partial s}{\partial X_2} D(X_2)$. Theorem 3.8.2 of this thesis shows how one can circumvent the condition that the divergence is 0 in the above theorem and show that the Cancellation Problem has an affirmative answer in two variables over an arbitrary \mathbb{Q} -algebra.

In three variables, the kernel of a locally nilpotent derivation on $k[X_1, X_2, X_3]$ was described by Miyanishi ([Miy85]). He proved the following theorem for an algebraically closed field of characteristic zero. Daigle remarked in [Dai97] that a straightforward use of [Kam75] then proves the general case.

Theorem 1.3.5. *Let D be a locally nilpotent derivation on $k[X_1, X_2, X_3]$. Then there are algebraically independent polynomials $f, g \in k[X_1, X_2, X_3]$ such that the kernel $k[X_1, X_2, X_3]^D$ of D equals $k[f, g]$. \square*

So the Cancellation Problem also has an affirmative answer in three variables over a field of characteristic zero. Theorem 4.5.4 generalises the result above, replacing the field k by a Dedekind domain containing \mathbb{Q} , thereby solving the Cancellation Problem over these rings in three variables. This generalised result is used by Corollary 4.5.5 to prove that the Cancellation Problem has an affirmative answer for a large class of derivations in four variables over a field of characteristic zero, including the triangular derivations.

1.4 Algorithm

Let k be a field of characteristic zero, let R be a k -domain and let A be an R -domain of finite type. Inspired by an algorithm of Tan to compute the invariant ring of certain G_a -actions on affine n -space ([Tan89]), Van den Essen gave an algorithm to compute generators of the kernel of a locally nilpotent R -derivation on A , provided that this kernel is of finite type over R ([Ess93]). Because the algorithm will be used further on, it will be explained here.

Definition 1.4.1. Let B be a R -subalgebra of A and $f \in B$. Define $B : f$ to be the R -subalgebra of A generated by the elements $g \in A$ such that $fg \in B$. Define $B : f^\infty$ to be the R -subalgebra of A generated by the elements $g \in A$ such that $f^m g \in B$ for some $m \in \mathbb{N}$.

The following lemma shows how such an algebra $B : f$ can actually be computed.

Lemma 1.4.2. *Assume that the finitely generated R -domain A is given as a quotient $A = R[X_1, \dots, X_n]/(f_1, \dots, f_m)$ for certain polynomials $f_1, \dots, f_m \in R[X] := R[X_1, \dots, X_n]$. Let B be a finitely generated R -subalgebra of A , say $B = R[\bar{g}_1, \dots, \bar{g}_s]$ for certain polynomials $g_1, \dots, g_s \in R[X]$. Here $\bar{}$ denotes taking residue classes modulo (f_1, \dots, f_m) . Let $f \in R[X]$ be a polynomial such that $\bar{f} \in B$.*

Let I be the ideal of $R[X, Y] := R[X_1, \dots, X_n, Y_1, \dots, Y_s]$ generated by f, f_1, \dots, f_m , and $Y_1 - g_1, \dots, Y_s - g_s$. Let J be the ideal $I \cap R[Y]$ of $R[Y]$, say $J = (h_1, \dots, h_t)$ for certain polynomials $h_1, \dots, h_t \in R[Y]$.

Then, for every $i = 1, \dots, t$, there exists a polynomial $k_i \in R[X]$ such that $h_i(\bar{g}_1, \dots, \bar{g}_s) = \bar{k}_i \bar{f}$. Furthermore, the R -algebra $B : \bar{f}$ is generated by the elements $\bar{g}_i, i = 1, \dots, s$, and $k_i, i = 1, \dots, t$.

Proof. Because $h_i \in I$, it can be written as

$$h_i = \tilde{k}_i f + \sum_{j=1}^m \tilde{a}_{ij} f_j + \sum \tilde{b}_{ij} (Y_j - g_j)$$

for certain polynomials $\tilde{k}_i, \tilde{a}_{ij}, \tilde{b}_{ij} \in R[X, Y]$. Since $h_i \in R[Y]$, substituting $Y_j := g_j$ for all j gives

$$h_i(g_1, \dots, g_s) = k_i f + \sum_{j=1}^m a_{ij} f_j$$

for certain polynomials $k_i, a_{ij} \in R[X]$. So $h_i(\bar{g}_1, \dots, \bar{g}_s) = \bar{k}_i \bar{f}$.

It is obvious that all \bar{g}_i and all \bar{k}_i are elements of $B : \bar{f}$. Conversely, take $g \in k[X]$ such that $\bar{g} \in B : \bar{f}$. Without loss of generality, assume that $\bar{f}\bar{g} \in B$. This means that $\bar{f}\bar{g} = h(\bar{g}_1, \dots, \bar{g}_s)$ for some polynomial $h \in R[Y] := R[Y_1, \dots, Y_s]$.

Now, note that $h \equiv h(g_1, \dots, g_s) \equiv fg \equiv 0 \pmod{I}$, so $h \in I \cap k[Y] = J$. This means that $h = \sum_{i=1}^s a_i h_i$ for certain polynomials $a_i \in k[Y]$. Consequently,

$$\bar{f}\bar{g} = \sum_{i=1}^s a_i(\bar{g}_1, \dots, \bar{g}_s) h_i(\bar{g}_1, \dots, \bar{g}_s)$$

and therefore $\bar{g} = \sum_{i=1}^s a_i(\bar{g}_1, \dots, \bar{g}_s) \bar{k}_i$. □

Remark 1.4.3. If the ring R is “nice enough”, then one can use Gröbner bases to actually find generators h_1, \dots, h_t of the ideal J . Just compute a Gröbner basis of the ideal I with respect to some admissible ordering in which all the X_i are larger than all the Y_j . The elements of the resulting Gröbner basis that are polynomials only in the variables Y_1, \dots, Y_s form a Gröbner basis of the ideal J .

One can also use Gröbner bases to actually compute the polynomials \tilde{k}_i , \tilde{a}_{ij} , and \tilde{b}_{ij} . So, if the ring is “nice enough” (for instance, if R is a computable field), then one can actually compute R -algebra generators of $B : \tilde{f}$. See for more information on Gröbner bases, for instance, [BW93].

Lemma 1.4.4. *Let A and B be as in the previous lemma. Now let I be the ideal of $R[X, Y]$ generated by the polynomials f_1, \dots, f_m , and $Y_1 - g_1, \dots, Y_s - g_s$. Then, for every $g \in R[X]$, $\bar{g} \in B$ if and only if there is an $h \in R[Y]$ such that $g \equiv h \pmod{I}$.*

Proof Easy. □

Remark 1.4.5. Again, if the ring R is “nice enough”, then one can use Gröbner bases to actually decide if such an h exists. Just take a Gröbner basis of I with respect to some admissible ordering in which all the X_i are larger than all the Y_j and find the normal form of g with respect to this Gröbner basis. If this normal form is a polynomial only in the variables Y_1, \dots, Y_s , then this is the h one was looking for. If this normal form is not a polynomial only in the variables Y_1, \dots, Y_s , then such a polynomial h does not exist.

Lemma 1.4.6. *Let B be an R -subalgebra of A and $f \in B$. Then*

1. *for all $g \in B$, $B : (fg) = (B : f) : g = (B : g) : f$ (so $B : f^m$ can denote both $B : (f^m)$ and the m -fold application of $: f$ to B);*
2. $B : f^\infty = \bigcup_{m \in \mathbb{N}} B : f^m$;
3. *if $B : f^\infty$ is a finitely generated R -algebra, then $B : f^\infty = B : f^m$ for some $m \in \mathbb{N}$;*
4. *if $B : f^m = B : f^{m+1}$ for some $m \in \mathbb{N}$, then $B : f^\infty = B : f^m$ and is a finitely generated R -algebra.*

Proof Easy. □

Proposition 1.4.7. *Let B be an R -subalgebra of A satisfying*

$$B \subseteq A^D \subseteq B[d^{-1}].$$

Then $A^D = B : d^\infty$.

Proof. \subseteq : Let $g \in A^D$. Then, by assumption, $g = f/d^m$ for some $f \in B$ and some $m \in \mathbb{N}$. So $d^m g = f \in B$, which implies that $g \in B : d^\infty$.

\supseteq : Let $g \in B : d^\infty$, say $m \in \mathbb{N}$ is such that $g \in B : d^m$. Without loss of generality, assume that $d^m g \in B$. Because $d^m \in A^D$ and $B \subseteq A^D$, $0 = D(d^m g) = d^m D(g)$. As A is a domain, it follows that $D(g) = 0$. \square

Assume that R is a k -algebra and that A itself is an R -algebra. Let D be a non-trivial locally nilpotent R -derivation on A . Let $p \in A$ be a preslice of D and write $d := D(p) \in A^D$. Now the derivation D can be extended in a unique (and obvious) way to a derivation on the localisation $A[d^{-1}]$. This extension will also be denoted by D . It is locally nilpotent and it has a slice, viz., $s := d^{-1}p$. So, by Theorem 1.1.3, $A[d^{-1}]^D = \varphi_{-s}(A[d^{-1}])$.

Now, say $A = R[a_1, \dots, a_n]$. Then $A[d^{-1}]^D = R[d^{-1}, \varphi_{-s}(a_1), \dots, \varphi_{-s}(a_n)]$ and multiplying the elements $\varphi_{-s}(a_i)$ by suitable powers of d , one finds elements $b_i \in A^D$ such that $A[d^{-1}]^D = R[d^{-1}, b_1, \dots, b_n]$. Now the R -subalgebra $B := R[b_1, \dots, b_n]$ of A satisfies $B \subseteq A^D \subseteq B[d^{-1}]$ and hence, by the previous proposition, $A^D = B : d^\infty$.

So, if the kernel of D is a finitely generated R -algebra, then it can be computed by successively calculating $B : f$, $B : f^2$, etc. If it turns out that $B : f^m = B : f^{m+1}$ for some $m \in \mathbb{N}$, then the full kernel has been computed. Also, if the kernel of D is not a finitely generated R -algebra, then it will never happen that $B : f^m = B : f^{m+1}$.

1.5 Polynomial automorphisms

Let R be a ring. Polynomial maps over R will play an important role in this thesis. Because this ring R can be rather nasty, it is worthwhile to provide a careful definition of this notion.

Polynomial maps

A *polynomial map* from R^n to R^m is defined to be a sequence $F := (f_1, \dots, f_m)$ of m polynomials in $R[X_1, \dots, X_n]$. The composition of a polynomial map $F = (f_1, \dots, f_m) : R^n \rightarrow R^m$ with a polynomial map $G = (g_1, \dots, g_l) : R^m \rightarrow R^l$ is defined by

$$G \circ F := (g_1(f_1, \dots, f_m), \dots, g_l(f_1, \dots, f_m)) : R^n \rightarrow R^l.$$

A polynomial map $F: R^n \rightarrow R^m$ can be interpreted as an actual function from R^n to R^m , sending $x \in R^n$ to $(f_1(x), \dots, f_m(x)) \in R^m$. This function will also be denoted by F . The reason for the awkward definition is that it is possible for two polynomial maps to represent the same function. This is shown in the following example.

Example 1.5.1. Consider the polynomial map $F := (X_1 - X_1^3): \mathbb{F}_3 \rightarrow \mathbb{F}_3$. Then $F(x) = 0$ for all $x \in \mathbb{F}_3$, so as a function it represents the zero-function. However, as a polynomial map it is different from the polynomial map $(0): \mathbb{F}_3 \rightarrow \mathbb{F}_3$.

As is well-known, one can associate to a polynomial map $F = (f_1, \dots, f_m): R^n \rightarrow R^m$ the R -algebra homomorphism $F^*: R[X_1, \dots, X_m] \rightarrow R[X_1, \dots, X_n]$ sending X_i to f_i . Conversely, every R -algebra homomorphism ψ from the polynomial ring $R[X_1, \dots, X_m]$ to the polynomial ring $R[X_1, \dots, X_n]$ determines a polynomial map $\psi^*: R^n \rightarrow R^m$, namely, $(\psi(X_1), \dots, \psi(X_m))$. For two composable polynomial maps F and G , $(F \circ G)^* = G^* \circ F^*$ and similarly for two composable R -algebra morphisms between polynomial rings over R .

Polynomial automorphisms

A polynomial map $F: R^n \rightarrow R^n$ is said to be a *polynomial automorphism* or an *invertible polynomial map* if there is a polynomial map $G: R^n \rightarrow R^n$ such that $F \circ G = G \circ F = (X_1, \dots, X_n)$. In other words, a polynomial map $F = (f_1, \dots, f_n): R^n \rightarrow R^n$ is invertible if and only if $R[f_1, \dots, f_n] = R[X]$. Also, the polynomial map F is invertible if and only if the corresponding R -algebra endomorphism $F^*: R[X] \rightarrow R[X]$ is in fact an automorphism.

The *Jacobian matrix* $(\frac{\partial f_i}{\partial X_j})_{i,j}$ of such a polynomial map is denoted by JF . Its determinant is sometimes called the *Jacobian determinant*. If F is invertible, then $\det JF \in R[X]^*$, so in R^* if R is a domain. The famous Jacobian Conjecture states the following.

Conjecture 1.5.2 (Jacobian Conjecture). *Let $F: \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a polynomial map with $\det JF \in \mathbb{C}^*$. Then F is invertible.*

For $n = 1$ this is true: the only polynomial maps from \mathbb{C} to \mathbb{C} with Jacobian determinant a non-zero constant in \mathbb{C} are the linear ones. For all other n , the conjecture is still open.

One can of course try to replace the field \mathbb{C} in the above conjecture by an arbitrary \mathbb{Q} -algebra R . There is, however, the following result.

Theorem 1.5.3. Assume that R is a \mathbb{Q} -algebra and take $n \in \mathbb{N}^*$. Let $F: R^n \rightarrow R^n$ be a polynomial map with Jacobian determinant in $R[X]^*$. If the Jacobian Conjecture is true for polynomial maps from \mathbb{C}^n to \mathbb{C}^n , then F is invertible. \square

Actually, the Jacobian Conjecture was first formulated for the case $R = \mathbb{Z}$ by Keller in [Kel39]. The following important theorem also appears in that paper for $R = \mathbb{Z}$.

Theorem 1.5.4 (Keller's Theorem). Let R be a domain with quotient field k . Let $F = (f_1, \dots, f_n): R^n \rightarrow R^n$ be a polynomial map with $\det JF \in R^*$. If $k(f_1, \dots, f_n) = k(X_1, \dots, X_n)$, then F is invertible. \square

Now consider a polynomial map $F = (f_1, \dots, f_n): R^n \rightarrow R^n$ with $F(0) = 0$. The Formal Inverse Function Theorem states that there is a uniquely determined $G = (g_1, \dots, g_n) \in R[[X]]^n$ with

$$F \circ G = (f_1(G), \dots, f_n(G)) = (X_1, \dots, X_n) = G \circ F.$$

The Jacobian Conjecture asserts that every component of G is in fact a polynomial. The power series G is called the *formal inverse* of F .

The coefficients of the formal inverse can be described using derivations on $R[X] := R[X_1, \dots, X_n]$. Define R -derivations $\partial/\partial F_i$ on $R[X]$ by

$$\begin{pmatrix} \frac{\partial}{\partial F_1} \\ \vdots \\ \frac{\partial}{\partial F_n} \end{pmatrix} := (JF^{-1})^T \begin{pmatrix} \frac{\partial}{\partial X_1} \\ \vdots \\ \frac{\partial}{\partial X_n} \end{pmatrix}.$$

One easily verifies that these derivations satisfy $\frac{\partial}{\partial F_i}(F_j) = \delta_{i,j}$ for all $i, j \in \{1, \dots, n\}$. According to Theorem 3.1.1 of [Ess00], the coefficients of the formal inverse can now be described as follows.

Proposition 1.5.5. Assume that R is a \mathbb{Q} -algebra and let

$$G = (g_1(Y), \dots, g_n(Y)) \in R[[Y]]^n$$

be the formal inverse of a polynomial map

$$F = (f_1(X), \dots, f_n(X)): R^n \rightarrow R^n$$

with $\det JF = 1$. Let D be the derivation $Y_1 \frac{\partial}{\partial F_1} + \dots + Y_n \frac{\partial}{\partial F_n}$ on the polynomial ring $R[X_1, \dots, X_n, Y_1, \dots, Y_n]$. Then

$$g_{i(d)} = \frac{1}{d!} D^d(X_i)[X_1 := 0, \dots, X_n := 0],$$

where $g_{i(d)}$ is the homogeneous part of degree d of the i th component of G . \square

Proposition 1.5.6. Assume that R is a \mathbb{Q} -algebra and let $G = (g_1, \dots, g_n) \in R[[X]]^n$ be the formal inverse of a polynomial map $F: R^n \rightarrow R^n$ with $\det JF = 1$. Using multi-index notation, write

$$g_i = \sum_{\alpha \in \mathbb{N}^n} c_{i,\alpha} X^\alpha$$

for certain $c_{i,\alpha} \in R$. Then

$$c_{i,\alpha} = \frac{1}{\alpha_1! \dots \alpha_n!} \left(\frac{\partial}{\partial F_1} \right)^{\alpha_1} \dots \left(\frac{\partial}{\partial F_n} \right)^{\alpha_n} (X_i) |_{X_1, \dots, X_n := 0}$$

for all $\alpha \in \mathbb{N}^n$. □

Coordinates

Let R be a ring and consider a polynomial automorphism $F = (f_1, \dots, f_n): R^n \rightarrow R^n$. The polynomials f_i are called *coordinates over R* . If the ring R is clear from the context, then they will simply be referred to as *coordinates*. More generally, for $k = 1, \dots, n$, the sequence (f_1, \dots, f_k) is called a *coordinate system over R* , or just a *coordinate system*.

Proposition 1.5.7. Assume that R is a \mathbb{Q} -algebra and let D be a locally nilpotent R -derivation on $R[X] := R[X_1, \dots, X_n]$ with a slice $s \in R[X]$. Then the following two statements are equivalent:

1. $R[X]^D \cong_R R^{[n-1]}$;
2. s is a coordinate.

Proof $1 \Rightarrow 2$: Consider an isomorphism $R[X_2, \dots, X_n] \rightarrow R[X]^D$ of R -algebras. By Theorem 1.1.2, this can be extended to an isomorphism $\psi: R[X] \rightarrow R[X] = R[X]^D[s]$ of R -algebras by sending X_1 to s . So $\psi^* = (s, \dots)$ is a polynomial automorphism over R , which shows that s is a coordinate.

$2 \Rightarrow 1$: By Theorem 1.1.2, $R[X]^D \cong_R R[X]/(s)$ and because s is a coordinate $R[X]/(s) \cong_R R[X]/(X_1) \cong_R R^{[n-1]}$. □

So this gives yet another way to formulate the Cancellation Problem (over \mathbb{C} , but in fact over every \mathbb{Q} -algebra).

Problem 1.5.8 (Cancellation Problem). Let D be a locally nilpotent derivation on $\mathbb{C}[X] := \mathbb{C}[X_1, \dots, X_n]$ with a slice $s \in \mathbb{C}[X]$. Is it then true that s is a coordinate?

1.6 Linearisation Problem

Let k be a field of characteristic zero. A final problem that will appear in Chapter 5 is the following linearisation problem for the cyclic group of order m .

Problem 1.6.1 (Linearisation Problem). *Let $F: k^n \rightarrow k^n$ be a polynomial automorphism and assume that $F^m = 1_{k^n}$ for some $m \in \mathbb{N}^*$. Does there always exist a polynomial automorphism $G: k^n \rightarrow k^n$ such that $G^{-1} \circ F \circ G$ is linear?*

In case $n = 2$ the answer to this question is affirmative and follows immediately from the fact that $\text{Aut}_k(k^2)$ is the amalgamated product of the affine subgroup and the subgroup of the De Jonquières transformations over their intersection (see [Kra95]). If $n \geq 3$ the problem remains open. However, one has the following relation between the Linearisation Problem and the Cancellation Problem.

Proposition 1.6.2. *If the Linearisation Problem has an affirmative answer (in dimension n) for all automorphisms of order 2, then the Cancellation Problem has an affirmative answer (in dimension n) as well.*

Proof Let $n \geq 2$ and let V be an algebraic variety over k . Assume that $\psi: V \times k \rightarrow k^n$ is an isomorphism of algebraic varieties. Let $F \in \text{Aut}_k(V \times k)$ be defined by $F(v, t) := (v, -t)$ for all $v \in V$ and $t \in k$ and take $G := \psi \circ F \circ \psi^{-1} \in \text{Aut}_k(k^n)$. Then obviously $G^2 = 1_{k^n}$ and $\text{Fix}(G) := \{x \in k^n \mid G(x) = x\} \cong_k V$.

Now if the Linearisation Problem has an affirmative answer for automorphisms of order two, then there exists an automorphism φ of k^n such that $\varphi^{-1} \circ G \circ \varphi = L$, which is a linear map. So $G = \varphi \circ L \circ \varphi^{-1}$, which implies that $\text{Fix}(G) \cong_k \text{Fix}(L)$. Because $\text{Fix}(L) \cong_k k^d$ for some d , it follows that $V \cong_k k^d$. Since $V \times k \cong_k k^n$, it even follows that $d = n - 1$. So $V \cong_k k^{n-1}$. \square

Chapter 2

Derivations and the Quillen-Suslin Theorem

This chapter considers several classes of locally nilpotent derivations for which one can see that the Cancellation Problem has an affirmative answer. Throughout this chapter, the notion of a Hermite ring and the Quillen-Suslin Theorem play an important role. The first section recalls them. The second section studies linear derivations and gives a counterexample to the Cancellation Problem for arbitrary rings. The third section is concerned with elementary derivations. It shows that the Cancellation Problem can be seen as a generalisation of the Quillen-Suslin Theorem. Finally, the fourth section exploits the Quillen-Suslin Theorem even further to create a larger class of locally nilpotent derivations for which the Cancellation Problem has an affirmative answer.

2.1 Hermite rings

Let A be a ring. A sequence (a_1, \dots, a_n) of elements of A is called a *unimodular row* if they generate the unit ideal of A . Such a sequence is said to be *extendible to an invertible square matrix over A* if there is an $M \in \text{Gl}(n, A)$ with the sequence as its first row. The ring A is called *Hermite* if every unimodular row over A can be extended to an invertible square matrix.

In 1955, Serre asked if every finitely generated projective $k[X_1, \dots, X_n]$ -module is free ([Ser55]). This question became known as Serre's Conjecture. There are many equivalent ways of formulating this conjecture. This thesis will mostly be concerned with the fourth formulation below: $k[X]$ is Hermite. Serre's Conjecture

was proven independently by Quillen ([Qui76]) and Suslin ([Sus76]) and is now commonly called the Quillen-Suslin Theorem.

Theorem 2.1.1 (Quillen-Suslin Theorem). *Let k be a field and let A be the polynomial ring $k[X_1, \dots, X_n]$ over k . Then*

1. *every finitely generated projective A -module is free;*
2. *every finitely generated stably free A -module is free;*
3. *every finitely generated stably free A -module of type 1 is free;*
4. *every unimodular row over A is extendible to an invertible square matrix, i.e., A is Hermite.* \square

For more information on unimodular rows, Hermite rings, and the Quillen-Suslin Theorem, see the classical account [Lam78] by Lam.

2.2 Linear derivations

Let A be a ring and consider the polynomial ring $A[X] := A[X_1, \dots, X_n]$. An A -derivation on $A[X]$ is called *linear* if it is of the form

$$D := a_1 \partial_{X_1} + \dots + a_n \partial_{X_n}$$

for certain $a_1, \dots, a_n \in A$. From now on, let D be such a derivation. Observe that $D^2(X_i) = 0$ for every i and that hence D is locally nilpotent.

Proposition 2.2.1. *The following three statements are equivalent:*

1. *D has a slice;*
2. *D has a linear slice;*
3. *$(a_1, \dots, a_n) = A$.*

Proof. $1 \Rightarrow 2$: Let $s \in A[X]$ be a slice of D . Write $s = s_{(0)} + s_{(1)} + \dots + s_{(d)}$, where $d := \deg s$ and each $s_{(i)}$ is the homogeneous part of s of degree i . Then

$$1 = D(s) = D(s_{(0)}) + D(s_{(1)}) + \dots + D(s_{(d)}).$$

Now note that $D(s_{(i)}) = 0$ or $\deg D(s_{(i)}) = i - 1$. Therefore $D(s_{(1)}) = 1$ (and $D(s_{(i)}) = 0$ for $i \neq 1$). So $s_{(1)}$ is a linear slice of D .

$2 \Rightarrow 3$: Suppose that $s \in A[X]$ is a linear slice of D , say $s = b_1X_1 + \cdots + b_nX_n$. Then $1 = D(s) = b_1a_1 + \cdots + b_na_n$, so $(a_1, \dots, a_n) = A$.

$3 \Rightarrow 1$: Suppose that $(a_1, \dots, a_n) = A$. Take $b_1, \dots, b_n \in A$ such that $b_1a_1 + \cdots + b_na_n = 1$ and take $s := b_1X_1 + \cdots + b_nX_n$. Then $D(s) = 1$. \square

Let k be a field of characteristic zero. From now on, assume that A is a k -domain of finite type. Let (a_1, \dots, a_n) be a unimodular row over A and let D be the derivation $a_1\partial_{X_1} + \cdots + a_n\partial_{X_n}$ on $A[X] := A[X_1, \dots, X_n]$. Then D has a (linear) slice $s \in A[X]$, say $s = b_1X_1 + \cdots + b_nX_n$.

Using Theorem 1.1.3, it is now easy to compute the kernel of D . Letting φ denote the exponential map of D on $A[X]$, it holds for every i that $\varphi_{-s}(X_i) = X_i - a_is$ and therefore $A[X]^D = A[X_1 - a_1s, \dots, X_n - a_ns]$. One can easily verify that this is in fact also true if the characteristic of k is not zero.

Lemma 2.2.2. *Let $F_1, \dots, F_{n-1} \in A[X]$ and assume that the kernel $A[X]^D$ of D equals $A[F_1, \dots, F_{n-1}]$. Take $f_i := F_{i(1)}$ (i.e., the linear part of F_i). Then $A[X]^D = A[f_1, \dots, f_{n-1}]$.*

Proof \subseteq : Assume, without loss of generality, that the polynomials F_1, \dots, F_{n-1} do not have a constant term.

Now consider $X_i - a_is \in A[X]^D = A[F_1, \dots, F_{n-1}]$. Then there is a polynomial $p(T_1, \dots, T_{n-1}) \in A[T_1, \dots, T_{n-1}]$ such that $X_i - a_is = p(F_1, \dots, F_{n-1})$. Then

$$\begin{aligned} X_i - a_is &= (p(F_1, \dots, F_{n-1}))_{(1)} && \text{(because } X_i - a_is \text{ is linear)} \\ &= (p_{(1)}(F_1, \dots, F_{n-1}))_{(1)} && \text{(because } F_1, \dots, F_{n-1} \text{ have} \\ &&& \text{no constant term)} \\ &= p_{(1)}(F_{1(1)}, \dots, F_{n-1(1)}) && \text{(because } p_{(1)} \text{ is linear)} \\ &= p_{(1)}(f_1, \dots, f_{n-1}) \in A[f_1, \dots, f_{n-1}] \end{aligned}$$

\supseteq : Because $F_i \in A[X]^D$ and the derivation D is homogeneous, every homogeneous part $F_{i(j)}$ of F_i is also in $A[X]^D$. In particular $f_i \in A[X]^D$. \square

Lemma 2.2.3. *Let $f_1, \dots, f_m \in A[X]$ be linear polynomials. Then*

$$A[f_1, \dots, f_m] \cap AX_1 \oplus \cdots \oplus AX_n = Af_1 + \cdots + Af_m.$$

In other words, every polynomial expression $p(f_1, \dots, f_m)$ in the f_i that is linear in the X_i is in fact an A -linear combination of the f_i .

Proof. \subseteq : Take $p(T_1, \dots, T_m) \in A[T_1, \dots, T_m]$ and let $g := p(f_1, \dots, f_m)$ be a polynomial expression in the f_i . Assume that g is in fact linear in the X_i . Then, using essentially the same argument as in the proof of the previous lemma,

$$\begin{aligned} g &= (p(f_1, \dots, f_m))_{(1)} \\ &= (p_{(1)}(f_1, \dots, f_m))_{(1)} \\ &= p_{(1)}(f_1, \dots, f_m) \in Af_1 + \dots + Af_n. \end{aligned}$$

\supseteq : This is obvious. \square

Lemma 2.2.4. *Let $f_1, \dots, f_{n-1} \in A[X]$ be linear polynomials and assume that $A[X]^D = A[f_1, \dots, f_{n-1}]$. Then*

$$As \oplus Af_1 \oplus \dots \oplus Af_{n-1} = AX_1 \oplus \dots \oplus AX_n.$$

In other words, every linear polynomial in $A[X]$ can be written in a unique way as an A -linear combination of s, f_1, \dots, f_{n-1} .

Proof. I will first show that $As + Af_1 + \dots + Af_{n-1} = AX_1 \oplus \dots \oplus AX_n$.

\subseteq : This is obvious.

\supseteq : Take $g \in AX_1 \oplus \dots \oplus AX_n$. Then $Dg \in A$ and therefore $D(g - (Dg)s) = Dg - (D^2g)s - (Dg)(Ds) = Dg - Dg = 0$. So,

$$\begin{aligned} g - (Dg)s &\in A[X]^D \cap AX_1 \oplus \dots \oplus AX_n \\ &= A[f_1, \dots, f_{n-1}] \cap AX_1 \oplus \dots \oplus AX_n \\ &= Af_1 + \dots + Af_{n-1} \quad (\text{by Lemma 2.2.3}) \end{aligned}$$

and hence $g \in As + Af_1 + \dots + Af_{n-1}$.

To see that $As + Af_1 + \dots + Af_{n-1}$ is in fact a direct sum, take scalars $\mu, \lambda_1, \dots, \lambda_{n-1} \in A$ and assume that $\mu s + \lambda_1 f_1 + \dots + \lambda_{n-1} f_{n-1} = 0$. Applying D to both sides yields $\mu = 0$, so

$$\lambda_1 f_1 + \dots + \lambda_{n-1} f_{n-1} = 0. \quad (2.1)$$

Now note that $A[f_1, \dots, f_{n-1}, s] = A[X]$ and hence (f_1, \dots, f_{n-1}, s) is a polynomial automorphism of $A[X]$. Associated to this automorphism are the derivations $\partial/\partial f_i$ and $\partial/\partial s$. Applying $\partial/\partial f_i$ to the equation (2.1) gives $\lambda_i = 0$, for each i . \square

Lemma 2.2.5. *Let $f_1, \dots, f_{n-1} \in A[X]$ be linear polynomials and assume that*

$$A[X]^D \cap AX_1 \oplus \dots \oplus AX_n = Af_1 + \dots + Af_{n-1}.$$

Then $A[X]^D = A[f_1, \dots, f_{n-1}]$.

Proof. By Theorem 1.1.3, the kernel $A[X]^D$ is generated by the linear polynomials $X_i - a_i s$, $i = 1, \dots, n$. By assumption, these polynomials are A -linear combinations of f_1, \dots, f_{n-1} . In particular, they are elements of $A[f_1, \dots, f_{n-1}]$. So $A[X]^D = A[f_1, \dots, f_{n-1}]$. \square

Proposition 2.2.6. $A^D \cong_A A[X_1, \dots, X_{n-1}]$ if and only if the unimodular row (a_1, \dots, a_n) can be extended to an invertible square matrix over A .

Proof. \Rightarrow : Assume that $A^D \cong_A A[X_1, \dots, X_{n-1}]$. This means that $A^D = A[F_1, \dots, F_{n-1}]$ for certain polynomials $F_1, \dots, F_{n-1} \in A[X]$. Then $F := (s, F_1, \dots, F_{n-1})$ is an invertible polynomial map over A and hence $\det JF \in A[X]^*$. Substituting $X_1 := 0, \dots, X_n := 0$ in the matrix JF gives an invertible square matrix over A extending the unimodular row (a_1, \dots, a_n) .

\Rightarrow : One could also prove this implication as follows. Assume that $A^D \cong_A A^{[n-1]}$, say $A^D = A[F_1, \dots, F_{n-1}]$ for certain polynomial $F_1, \dots, F_n \in A[X]$. By Lemma 2.2.2, A^D even equals $A[f_1, \dots, f_{n-1}]$ for certain linear polynomials $f_1, \dots, f_{n-1} \in A[X]$. Now Lemma 2.2.4 implies that

$$As \oplus Af_1 \oplus \dots \oplus Af_{n-1} = AX_1 \oplus \dots \oplus AX_n,$$

say $f_i = \lambda_{i1}X_1 + \dots + \lambda_{in}X_n$ for certain $\lambda_{ij} \in A$ (and $s = b_1X_1 + \dots + b_nX_n$). This is an equality of free A -modules of rank n and the base transformation matrix is

$$\begin{pmatrix} b_1 & \lambda_{11} & \dots & \lambda_{n-1,1} \\ \vdots & \vdots & & \vdots \\ b_n & \lambda_{1n} & \dots & \lambda_{n-1,n} \end{pmatrix}.$$

The inverse of this matrix is an invertible square matrix over A extending the unimodular row (a_1, \dots, a_n) .

\Leftarrow : Let M be an extension of (a_1, \dots, a_n) to an invertible square matrix over A . It is possible to assume that $\det M = 1$. Consider its inverse M^{-1} , say $M^{-1} = (b_{ij})_{i,j}$, and define $f_j := b_{1j}X_1 + \dots + b_{nj}X_n$. Then $D(f_1) = 1$ and $D(f_2) = \dots = D(f_n) = 0$. Furthermore, every linear polynomial f over A that satisfies $D(f) = 0$ is a linear combination of f_2, \dots, f_n , i.e., $A[X]^D \cap AX_1 \oplus \dots \oplus AX_n = Af_2 + \dots + Af_n$. Lemma 2.2.5 now implies that $A[X]^D = A[f_2, \dots, f_n] \cong_A A^{[n-1]}$. \square

Corollary 2.2.7. *The Cancellation Problem has an affirmative answer for linear derivations over A if and only if A is Hermite.* \square

Example 2.2.8. 1. Take $A := \mathbb{R}[T_1, T_2, T_3]/(T_1^2 + T_2^2 + T_3^2 - 1)$. Then it is well-known that the unimodular row $(\bar{T}_1, \bar{T}_2, \bar{T}_3)$ cannot be extended to an invertible square matrix. So the linear derivation $D := \bar{T}_1\partial_X + \bar{T}_2\partial_Y + \bar{T}_3\partial_Z$ on the polynomial ring $A[X, Y, Z]$ has a slice, but $A[X, Y, Z]^D \not\cong_A A^{[2]}$. This example is even more interesting, since a short argument from Hochster ([Hoc72]) shows that $A[X, Y, Z]^D \not\cong_{\mathbb{R}} A^{[2]}$ and even that $A[X, Y, Z]^D \not\cong_A A^{[2]}$.

2. Over the complex numbers, one can consider the ring A defined by $A := \mathbb{C}[S_1, S_2, S_3, T_1, T_2, T_3]/(S_1T_1 + S_2T_2 + S_3T_3 - 1)$. It was shown by Raynaud in [Ray68] and by Suslin in [Sus82] that the unimodular row $(\bar{S}_1, \bar{S}_2, \bar{S}_3)$ cannot be extended to an invertible square matrix. So the derivation $D := \bar{S}_1\partial_X + \bar{S}_2\partial_Y + \bar{S}_3\partial_Z$ has a slice as well, but $A[X, Y, Z]^D \not\cong_A A^{[2]}$.

2.3 Elementary derivations

Let A be a ring and $n \in \mathbb{N}^*$. An A -derivation D on $A[X_1, \dots, X_n]$ is called *elementary* if it is of the form

$$D = f_1(X_{k+1}, \dots, X_n)\partial_{X_1} + \dots + f_k(X_{k+1}, \dots, X_n)\partial_{X_k}$$

for some $k \in \{1, \dots, n\}$ and some polynomials $f_1, \dots, f_k \in A[X_{k+1}, \dots, X_n]$.

The kernels of elementary derivations over a field k of characteristic 0 were first systematically studied by Van den Essen and Janssen in [EJ95]. As was already mentioned in Chapter 1, for $n = 2$ each locally nilpotent derivation on $k[X_1, X_2]$ is conjugate to an elementary derivation of the form $f(X_2)\partial_{X_1}$. In [EJ95], it is argued that this is not the case for $n \geq 4$. For consider the derivation $D := (1 + X_3^2)\partial_{X_1} + X_3\partial_{X_2} + X_4\partial_{X_3}$. This derivation is locally nilpotent and fixed point free. Suppose it is conjugate to an elementary derivation. This elementary derivation would then also be fixed point free and, by Proposition 2.2.1, have a slice. This would mean that the derivation D itself would have a slice, but the paper [DFG94] by Deveney, Finston, and Gehrke shows that it does not. Later, in Example 3.3.8 it will be shown in a different way that this derivation does not have a slice. For $n = 3$, Bass has shown in [Bas84] that the derivation $D := (X_1X_3 + X_2^2)(X_1\partial_{X_2} + 2X_2\partial_{X_3})$ on $k[X_1, X_2, X_3]$ is locally nilpotent and not triangulable. In particular, D is not conjugate to an elementary derivation.

Another example of an important elementary derivation is $D := X^3\partial_S + Y^3\partial_T + Z^3\partial_U + (XYZ)^2\partial_V$ on the polynomial ring $k[S, T, U, V, X, Y, Z]$. This derivation was constructed by Deveney and Finston in [DF94] and is based on Robert's counterexample to Hilbert's 14th Problem. It is an example of a locally nilpotent derivation whose kernel is not a finitely generated k -algebra.

As a final example, consider the so-called *Weitzenböck derivation*. This is the derivation $D := T_1\partial_{X_1} + \cdots + T_n\partial_{X_n}$ on $k[X_1, \dots, X_n, T_1, \dots, T_n]$. One can easily see that the elements $X_iT_j - X_jT_i$ and X_i are in the kernel of D . For $n \leq 4$ these elements generate the kernel of D , but for larger n it is not known if that is the case. See [Now94] for more information.

So there are lots of complicated and nasty elementary derivations. Regarding the Cancellation Problem, however, they behave very nicely. By the Quillen-Suslin Theorem, a polynomial ring over a field is Hermite. This gives the following result, which was already present in [EJ95].

Proposition 2.3.1. *The Cancellation Problem has an affirmative answer for all elementary derivations over a field k of characteristic 0.*

Proof. Consider an elementary derivation

$$D := f_1(Y_1, \dots, Y_m)\partial_{X_1} + \cdots + f_n(Y_1, \dots, Y_m)\partial_{X_n}$$

on $k[Y, X]$. This is also a linear $k[Y]$ -derivation on the polynomial ring $k[Y][X]$ in n variables over $k[Y]$. By the Quillen-Suslin Theorem, $k[Y]$ is Hermite and hence Corollary 2.2.7 implies that $k[Y][X]^D = k[Y][g_1, \dots, g_{n-1}] = k[Y, g_1, \dots, g_{n-1}]$ for $n - 1$ polynomials $g_i \in k[Y][X]$. \square

2.4 $\mathcal{T}(n, A)$

The Quillen-Suslin Theorem can be exploited even further to investigate the Cancellation Problem. Using the theorem, this section constructs a class of locally nilpotent \mathbb{C} -derivations on $\mathbb{C}[X] := \mathbb{C}[X_1, \dots, X_n]$ for which the Cancellation Problem can be answered affirmatively.

The construction is based on the following observation. Suppose for a moment that the Cancellation Problem is true. Then every locally nilpotent \mathbb{C} -derivation on $\mathbb{C}[X]$ with a slice has a coordinate in its kernel (namely the f_i from the formulation of the Cancellation Problem). Therefore, D can be written as $D = \varphi \circ \tilde{D} \circ \varphi^{-1}$ for some polynomial automorphism φ and some derivation \tilde{D} with $\tilde{D}(X_n) = 0$. This \tilde{D} can be considered as a derivation over $\mathbb{C}(X_n)$ in $n - 1$ variables. It has a slice

and hence should have a coordinate in $\mathbb{C}(X_n)[X_1, \dots, X_{n-1}]$ in its kernel. So \tilde{D} can be written in a similar way as D .

Now, obviously, the Cancellation Problem has not yet been solved, but the idea is to construct a class of derivations over a ring A with a (linear) coordinate in their kernels out of a class of derivations over the ring $A[X_n]$ which is likewise constructed. The actual construction in Definition 2.4.2 bears a great deal of resemblance to the construction of $\mathcal{H}(n, A)$ by Van den Essen and Hubbers in [EH96]. See also [Hub98] and [EH97].

Notation 2.4.1. Let A be a ring and $n \in \mathbb{N}^*$. If M is an $n \times n$ matrix over A , it can be considered as a polynomial map $M: A^n \rightarrow A^n$. The corresponding endomorphism of $A[X] := A[X_1, \dots, X_n]$ is denoted by M^* . Similarly, if $c \in A^n$, then the translation $T_c: A^n \rightarrow A^n$ given by $x \mapsto x + c$, has a corresponding map $T_c^*: A[X_1, \dots, X_n] \rightarrow A[X_1, \dots, X_n]$ of algebras given by $T_c^*(X_i) = X_i + c_i$, for all $i \in \{1, \dots, n\}$.

Definition 2.4.2. For every ring A and for every $n \in \mathbb{N}^*$ define the collection $\mathcal{T}(n, A) \subseteq \text{Der}_A(A[X_1, \dots, X_n])$ inductively by

$$\mathcal{T}(1, A) := \{a\partial_{X_1} \mid a \in A\},$$

and, for $n \geq 2$,

$$\mathcal{T}(n, A) := \{M^* \circ D \circ (M^{\text{ad}})^* \mid M \in \text{Mat}(n, A), D \in \mathcal{T}(n-1, A[X_n])\}.$$

Example 2.4.3. Let $n = 2$ and take $M = \begin{pmatrix} b_1 & b_2 \\ a_1 & a_2 \end{pmatrix} \in \text{Mat}(2, A)$. Then

$$M^*(X_1) = b_1X_1 + b_2X_2,$$

$$M^*(X_2) = a_1X_1 + a_2X_2,$$

$$M^{\text{ad}} = \begin{pmatrix} a_2 & -b_2 \\ -a_1 & b_1 \end{pmatrix}, \text{ and}$$

$$(M^{\text{ad}})^*(X_1) = a_2X_1 - b_2X_2,$$

$$(M^{\text{ad}})^*(X_2) = -a_1X_1 + b_1X_2.$$

The general form of an element from $\mathcal{T}(1, A[X_2])$ is $\tilde{D} = f(X_2)\partial_{X_1}$, with $f(X_2)$ an element of $A[X_2]$. Now consider the derivation $D := M^* \circ \tilde{D} \circ (M^{\text{ad}})^*$ and compute $D(X_1)$ and $D(X_2)$:

$$\begin{aligned} D(X_1) &= (M^* \circ \tilde{D})(a_2X_1 - b_2X_2) \\ &= M^*(a_2f(X_2)) \\ &= a_2f(a_1X_1 + a_2X_2) \end{aligned}$$

and similarly

$$D(X_2) = -a_1 f(a_1 X_1 + a_2 X_2).$$

Therefore

$$D = M^* \circ \tilde{D} \circ (M^{\text{ad}})^* = f(a_1 X_1 + a_2 X_2)(a_2 \partial_{X_1} - a_1 \partial_{X_2})$$

is the general form of an element of $\mathcal{T}(2, A)$.

Example 2.4.4. Let $n = 3$ and take $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} \in \text{Mat}(3, A)$. Then

$$M^{\text{ad}} = \begin{pmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ -\lambda_1 & -\lambda_2 & 1 \end{pmatrix}.$$

It follows that from the previous example that the general form of an element of $\mathcal{T}(2, A[X_3])$ is

$$\tilde{D} = f(a_1(X_3)X_1 + a_2(X_3)X_2)(a_2(X_3)\partial_{X_1} - a_1(X_3)\partial_{X_2}).$$

Now consider $D := M^* \circ \tilde{D} \circ (M^{\text{ad}})^*$. Then

$$\begin{aligned} D(X_1) &= (M^* \circ \tilde{D})(\lambda_3 X_1) \\ &= M^*(\lambda_3 f(a_1(X_3)X_1 + a_2(X_3)X_2) a_2(X_3)) \\ &= \lambda_3 f(a_1(\langle \lambda, X \rangle)X_1 + a_2(\langle \lambda, X \rangle)X_2) a_2(\langle \lambda, X \rangle), \end{aligned}$$

where $\langle \lambda, X \rangle := \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3$,

$$\begin{aligned} D(X_2) &= (M^* \circ \tilde{D})(\lambda_3 X_2) \\ &= M^*(-\lambda_3 f(a_1(X_3)X_1 + a_2(X_3)X_2) a_1(X_3)) \\ &= -\lambda_3 f(a_1(\langle \lambda, X \rangle)X_1 + a_2(\langle \lambda, X \rangle)X_2) a_1(\langle \lambda, X \rangle), \end{aligned}$$

and

$$\begin{aligned} D(X_3) &= (M^* \circ \tilde{D})(-\lambda_1 X_1 - \lambda_2 X_2 + X_3) \\ &= M^*(f(a_1(X_3)X_1 + a_2(X_3)X_2) (-\lambda_1 a_2(X_3) + \lambda_2 a_1(X_3))) \\ &= f(a_1(\langle \lambda, X \rangle)X_1 + a_2(\langle \lambda, X \rangle)X_2) (-\lambda_1 a_2(\langle \lambda, X \rangle) + \lambda_2 a_1(\langle \lambda, X \rangle)). \end{aligned}$$

Therefore, the derivations of the form

$$\begin{aligned}
 D &= f(a_1(\langle \lambda, X \rangle)X_1 + a_2(\langle \lambda, X \rangle)X_2) \cdot \\
 &\quad (\lambda_3 a_2(\langle \lambda, X \rangle)\partial_{X_1} - \lambda_3 a_1(\langle \lambda, X \rangle)\partial_{X_2} + \\
 &\quad + (\lambda_2 a_1(\langle \lambda, X \rangle) - \lambda_1 a_2(\langle \lambda, X \rangle))\partial_{X_3}) \\
 &= f(a_1(\langle \lambda, X \rangle)X_1 + a_2(\langle \lambda, X \rangle)X_2) \cdot \\
 &\quad (a_2(\langle \lambda, X \rangle)(\lambda_3 \partial_{X_1} - \lambda_1 \partial_{X_3}) - a_1(\langle \lambda, X \rangle)(\lambda_3 \partial_{X_2} - \lambda_2 \partial_{X_3}))
 \end{aligned}$$

are elements of $\mathcal{T}(3, A)$.

Now first of all, the derivations in $\mathcal{T}(n, A)$ are all locally nilpotent. In fact, they even satisfy the following, stronger, property.

Proposition 2.4.5. *For every $n \in \mathbb{N}^*$, for every ring A , and for every $D \in \mathcal{T}(n, A)$, $D^2(X_i) = 0$ for all $i \in \{1, \dots, n\}$.*

Proof By induction on n it will follow that for every $n \in \mathbb{N}^*$, for every ring A , for every $\lambda \in A$, and for every $D \in \mathcal{T}(n, A)$, $(D \circ (\lambda I)^* \circ D)(X_i) = 0$ for all $i \in \{1, \dots, n\}$. The proposition then follows by taking $\lambda := 1$.

$n = 1$: Easy.

$n > 1$: Let A be any ring and $\lambda \in A$. Let $D \in \mathcal{T}(n, A)$, say $D = M^* \circ \tilde{D} \circ (M^{\text{ad}})^*$ with $M \in \text{Mat}(n, A)$ and $\tilde{D} \in \mathcal{T}(n-1, A[X_n])$. Then

$$\begin{aligned}
 D \circ (\lambda I)^* \circ D &= M^* \circ \tilde{D} \circ (M^{\text{ad}})^* \circ (\lambda I)^* \circ M^* \circ \tilde{D} \circ (M^{\text{ad}})^* \\
 &= M^* \circ \tilde{D} \circ (\lambda \mu I)^* \circ \tilde{D} \circ (M^{\text{ad}})^*,
 \end{aligned}$$

where $\mu := \det(M) \in A$. Since $M^*(X_i)$ is an A -linear combination of X_1, \dots, X_n , it now follows from the induction hypothesis that $(\tilde{D} \circ (\lambda I)^* \circ \tilde{D})(X_i) = 0$. \square

The goal of this section is to prove that the Cancellation Problem has an affirmative answer for the derivations in $\mathcal{T}(n, A)$. In order to do so, consider the following subset $\mathcal{T}^s(n, A)$ of $\mathcal{T}(n, A)$.

Definition 2.4.6. For every ring A and for every $n \in \mathbb{N}^*$ define the collection $\mathcal{T}^s(n, A) \subseteq \text{Der}_A(A[X_1, \dots, X_n])$ inductively by

$$\mathcal{T}^s(1, A) := \{a\partial_{X_1} \mid a \in A^*\},$$

and, for $n \geq 2$,

$$\mathcal{T}^s(n, A) := \{M^* \circ D \circ (M^{\text{ad}})^* \mid M \in \text{Sl}(n, A), D \in \mathcal{T}^s(n-1, A[X_n])\}.$$

The bulk of this section is devoted to the proof of the following theorem. It characterises exactly when an element from $\mathcal{T}(n, A)$ has a slice. The main result (Corollary 2.4.14) is an immediate consequence.

Theorem 2.4.7. *Assume that A is a domain and that A and all polynomial rings $A[X_1, \dots, X_k]$, $k \in \mathbb{N}^*$, are Hermite. Let $D \in \mathcal{T}(n, A)$. Then*

$$D \text{ has a slice} \iff D \in \mathcal{T}^s(n, A).$$

In order to prove this theorem, five lemmas are needed. The first three explain more about the structure of $\mathcal{T}(n, A)$ and $\mathcal{T}^s(n, A)$; the last two address specific technical issues needed in the proof of the above theorem.

Notation 2.4.8. Let $\varphi: A \rightarrow B$ be a ring homomorphism and let D be an A -derivation on $A[X] := A[X_1, \dots, X_n]$, say $D = f_1(X)\partial_{X_1} + \dots + f_n(X)\partial_{X_n}$. Then $\varphi_*(D)$ denotes the induced B -derivation on $B[X_1, \dots, X_n]$ that one gets by applying φ to the coefficients of all monomials appearing in the polynomials f_1, \dots, f_n .

Lemma 2.4.9. *Let $\varphi: A \rightarrow B$ be a ring homomorphism and let D be an A -derivation on $A[X_1, \dots, X_n]$. Then*

$$D \in \mathcal{T}(n, A) \implies \varphi_*(D) \in \mathcal{T}(n, B)$$

and

$$D \in \mathcal{T}^s(n, A) \implies \varphi_*(D) \in \mathcal{T}^s(n, B).$$

Proof. By an easy induction on n . □

Lemma 2.4.10. *Let $D \in \mathcal{T}(n, A)$ and $M \in \text{Mat}(n, A)$. Then $M^* \circ D \circ (M^{ad})^* \in \mathcal{T}(n, A)$. Also, if $D \in \mathcal{T}^s(n, A)$ and $M \in \text{Sl}(n, A)$, then $M^* \circ D \circ (M^{ad})^* \in \mathcal{T}^s(n, A)$.*

Proof. Easy. □

Lemma 2.4.11. *Let $c \in A^n$ and $D \in \text{Der}_A(A[X_1, \dots, X_n])$. Then*

$$D \in \mathcal{T}(n, A) \implies T_c^* \circ D \circ T_{-c}^* \in \mathcal{T}(n, A)$$

and

$$D \in \mathcal{T}^s(n, A) \implies T_c^* \circ D \circ T_{-c}^* \in \mathcal{T}^s(n, A).$$

Proof. By induction on n .

$n = 1$: If $D \in \mathcal{T}(1, A)$, then $D = a\partial_{X_1}$ for an $a \in A$. Then $T_c^* \circ D \circ T_{-c}^* = D \in \mathcal{T}(1, A)$.

A similar argument holds for the $\mathcal{T}^s(1, A)$ -case.

$n > 1$: Assume that $D \in \mathcal{T}(n, A)$, say $D = M^* \circ \tilde{D} \circ (M^{\text{ad}})^*$ with $\tilde{D} \in \mathcal{T}(n-1, A[X_n])$ and $M \in \text{Mat}(n, A)$. Say $M^{\text{ad}} = (\mu_{i,j})_{i,j=1,\dots,n}$ with $\mu_{1,1}, \dots, \mu_{n,n} \in A$ and $Mc = (v_1, \dots, v_n)$ with $v_1, \dots, v_n \in A$. Then, for all $i \in \{1, \dots, n\}$,

$$\begin{aligned} (\tilde{D} \circ (M^{\text{ad}})^* \circ T_{-c}^*)(X_i) &= (\tilde{D} \circ (M^{\text{ad}})^*)(X_i - c_i) \\ &= \tilde{D}(\mu_{i,1}X_1 + \dots + \mu_{i,n}X_n - c_i) \\ &= \mu_{i,1}\tilde{D}(X_1) + \dots + \mu_{i,n}\tilde{D}(X_n) \end{aligned}$$

and also

$$\begin{aligned} (\tilde{D} \circ T_{-Mc}^* \circ (M^{\text{ad}})^*)(X_i) &= (\tilde{D} \circ T_{-Mc}^*)(\mu_{i,1}X_1 + \dots + \mu_{i,n}X_n) \\ &= \tilde{D}(\mu_{i,1}(X_1 - v_1) + \dots + \mu_{i,n}(X_n - v_n)) \\ &= \mu_{i,1}\tilde{D}(X_1) + \dots + \mu_{i,n}\tilde{D}(X_n). \end{aligned}$$

Therefore, for all $i \in \{1, \dots, n\}$,

$$\begin{aligned} T_c^* \circ D \circ T_{-c}^*(X_i) &= T_c^* \circ M^* \circ \tilde{D} \circ (M^{\text{ad}})^* \circ T_{-c}^*(X_i) \\ &= M^* \circ T_{Mc}^* \circ \tilde{D} \circ (M^{\text{ad}})^* \circ T_{-c}^*(X_i) \\ &= M^* \circ T_{Mc}^* \circ \tilde{D} \circ T_{-Mc}^* \circ (M^{\text{ad}})^*(X_i), \end{aligned}$$

which implies

$$T_c^* \circ D \circ T_{-c}^* = M^* \circ T_{Mc}^* \circ \tilde{D} \circ T_{-Mc}^* \circ (M^{\text{ad}})^*.$$

Now the only thing left to show is that $T_{Mc}^* \circ \tilde{D} \circ T_{-Mc}^*$ is an element of $\mathcal{T}(n-1, A[X_n])$, because then the derivation $M^* \circ T_{Mc}^* \circ \tilde{D} \circ T_{-Mc}^* \circ (M^{\text{ad}})^* \circ T_{-c}^*$ is, by definition, an element of $\mathcal{T}(n, A)$.

However, Mc is a vector of length n over A (and not of length $n-1$ over $A[X_n]$), so the induction hypothesis cannot be applied immediately. To resolve this, write $v' = (v_1, \dots, v_{n-1}) \in A^{n-1} \subseteq A[X_n]^{n-1}$ and $v'' = (0, \dots, 0, v_n) \in A^n$. Note that $T_{v''}^*$ and T_{Mc}^* are A -automorphisms of $A[X_1, \dots, X_n]$ and that $T_{v'}^*$, which actually is an $A[X_n]$ -automorphism of

$A[X_n][X_1, \dots, X_{n-1}]$, can also be considered as one. Then $T_{Mc}^* = T_{v'}^* \circ T_{v''}^*$ and also $T_{v''}^* \circ \tilde{D} \circ T_{-v''}^* = \varphi_*(\tilde{D})$, where $\varphi_*(\tilde{D})$ is the $A[X_n]$ -derivation on $A[X_n][X_1, \dots, X_{n-1}] = A[X_1, \dots, X_n]$ induced by the homomorphism $\varphi: A[X_n] \rightarrow A[X_n]$ sending X_n to $X_n + v_n$. So

$$\begin{aligned} T_{Mc}^* \circ \tilde{D} \circ T_{-Mc}^* &= T_{v'}^* \circ T_{v''}^* \circ \tilde{D} \circ T_{-v''}^* \circ T_{-v'}^* \\ &= T_{v'}^* \circ \varphi_*(\tilde{D}) \circ T_{-v'}^*. \end{aligned}$$

By Lemma 2.4.9, $\varphi_*(\tilde{D}) \in \mathcal{T}(n-1, A[X_n])$ and hence, by the induction hypothesis, $T_{Mc}^* \circ \tilde{D} \circ T_{-Mc}^* = T_{v'}^* \circ \varphi_*(\tilde{D}) \circ T_{-v'}^* \in \mathcal{T}(n-1, A[X_n])$. This proves the claim.

Exactly the same argument holds for the $\mathcal{T}^s(n, A)$ -case. \square

Lemma 2.4.12. *Assume that A is a domain. Let $M \in \text{Mat}(n, A)$ with $\det(M) \neq 0$, let $a \in A^*$ and let $F \in \text{Aut}_A(A[X_1, \dots, X_n])$. Let D be the derivation $M^* \circ F \circ a\partial_{X_1} \circ F^{-1} \circ (M^{\text{ad}})^*$. Assume that the constant part $F_{(0)}$ of F equals 0, that the linear part $F_{(1)}$ of F satisfies $F_{(1)}(X_n) = X_n$ and that D has a slice. Then the last row of M is unimodular.*

Proof Let $s = s(X_1, \dots, X_n) \in A[X_1, \dots, X_n]$ be a slice of D . Now, because $\det(M) \neq 0$, the map $M^* \circ F$ can be considered as an automorphism over $Q(A)$. Therefore the derivation $a\partial_{X_1}$ has a slice as well, viz. $(F^{-1} \circ (M^{\text{ad}})^*)(s)$, and hence

$$(F^{-1} \circ (M^{\text{ad}})^*)(s) = a^{-1}X_1 + p(X_2, \dots, X_n)$$

for a certain polynomial $p(X_2, \dots, X_n) \in A[X_2, \dots, X_n]$. Moving F over to the other side of the equation and applying M^* to both sides yields

$$\begin{aligned} s(dX_1, \dots, dX_n) &= (M^* \circ (M^{\text{ad}})^*)(s) \\ &= (M^* \circ F)(a^{-1}X_1 + p(X_2, \dots, X_n)), \end{aligned}$$

where $d := \det(M)$.

The linear part of the left hand side is divisible by d and hence the linear part of the right hand side is divisible by d . Write $F_{(1)}$ for the linear part of F , say $F_{(1)} = \Phi^*$, with $\Phi \in \text{Mat}(n, A)$. Note that because $F_{(1)}(X_n) = X_n$, the last row of the matrix Φ equals $(0 \ \dots \ 0 \ 1)$. Also, write $p_{(1)}$ for the linear part of p , say $p_{(1)} = c_2X_2 + \dots + c_nX_n$ for certain $c_2, \dots, c_n \in A$.

It is possible to assume, without loss of generality, that $s(0) = 0$. Furthermore, F has no constant part and therefore the linear part of the right hand side equals

$$\begin{aligned} (M^* \circ F_{(1)})(a^{-1}X_1 + p_{(1)}) &= (M^* \circ \Phi^*)(a^{-1}X_1 + c_2X_2 + \cdots + c_nX_n) \\ &= (\Phi M)^*((a^{-1} \ c_2 \ \dots \ c_n)X) \\ &= (a^{-1} \ c_2 \ \dots \ c_n)\Phi MX. \end{aligned}$$

Because this is divisible by d , every component of $(a^{-1} \ c_2 \ \dots \ c_n)\Phi M$ is divisible by d .

So, there is an equality

$$d(b_1 \ \dots \ b_n) = (a^{-1} \ c_2 \ \dots \ c_n)\Phi M$$

between row vectors, with $b_1, \dots, b_n \in A$. Rewriting this equation gives

$$\begin{aligned} d(b_1 \ \dots \ b_n)M^{\text{ad}} &= (a^{-1} \ c_2 \ \dots \ c_n)\Phi MM^{\text{ad}} \\ &= (a^{-1} \ c_2 \ \dots \ c_n)\Phi d, \end{aligned}$$

and therefore also

$$a(b_1 \ \dots \ b_n)M^{\text{ad}}\Phi^{-1} = (1 \ ac_2 \ \dots \ ac_n).$$

Since the last row of Φ equals $(0 \ \dots \ 0 \ 1)$, the last row of Φ^{-1} is $(0 \ \dots \ 0 \ 1)$. Computing the first component of the product on the left hand sides gives

$$\begin{aligned} 1 &= a \sum_{i=1}^n \sum_{j=1}^n b_i M_{i,j}^{\text{ad}} \Phi_{j,1}^{-1} \\ &= a \sum_{i=1}^n \sum_{j=1}^{n-1} b_i M_{i,j}^{\text{ad}} \Phi_{j,1}^{-1} \\ &= a \sum_{i=1}^n \sum_{j=1}^{n-1} b_i ((-1)^{ij} \text{minor}_{j,i}(M)) \Phi_{j,1}^{-1}. \end{aligned}$$

As $\text{minor}_{j,i}(M)$ is a linear combination of every row of the matrix M except the j th, and because the $j = n$ term has dropped out of the sum, the whole sum is a linear combination of the last row of M . Hence 1 is a linear combination of the last row, which means that this last row is unimodular. \square

Lemma 2.4.13. *Let $\tilde{D} \in \mathcal{T}(n-1, A[X_n])$ and $M \in \text{Mat}(n, A)$ with $\det(M) \neq 0$. Let D be the derivation $M^* \circ \tilde{D} \circ (M^{\text{ad}})^*$. Assume that the last row of M equals $(0 \ \dots \ 0 \ 1)$. Then $D \in \mathcal{T}(n-1, A[X_n])$ as well.*

Proof Since the last row of M equals $(0 \ \dots \ 0 \ 1)$, it is possible to decompose M as $M = E\tilde{M}$, where E and \tilde{M} are of the form

$$E = \begin{pmatrix} 1 & & & a_1 \\ & \ddots & & \vdots \\ & & 1 & a_{n-1} \\ 0 & \dots & 0 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{M} = \begin{pmatrix} * & \dots & * & 0 \\ \vdots & & \vdots & \vdots \\ * & \dots & * & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix},$$

respectively.

Take $c := (a_1 X_n, \dots, a_{n-1} X_n) \in A[X_n]^{n-1}$. Then $E^* = T_c^*$ and hence

$$\begin{aligned} D &= M^* \circ \tilde{D} \circ (M^{\text{ad}})^* = (E\tilde{M})^* \circ \tilde{D} \circ ((E\tilde{M})^{\text{ad}})^* \\ &= (\tilde{M})^* \circ E^* \circ \tilde{D} \circ (E^{-1})^* \circ (\tilde{M}^{\text{ad}})^* \\ &= (\tilde{M})^* \circ T_c^* \circ \tilde{D} \circ T_{-c}^* \circ (\tilde{M}^{\text{ad}})^*, \end{aligned}$$

Note that, by Lemma 2.4.11, $T_c^* \circ \tilde{D} \circ T_{-c}^* \in \mathcal{T}(n-1, A[X_n])$. Furthermore, note that \tilde{M} , which is in fact an $n \times n$ matrix over A , can as well be considered to be an $(n-1) \times (n-1)$ -matrix over $A[X_n]$. Hence, Lemma 2.4.10 yields that D is an element of $\mathcal{T}(n-1, A[X_n])$. \square

Proof of Theorem 2.4.7. \Leftarrow : Obvious.

\Rightarrow : By induction on n . The case $n = 1$ is obvious, so assume that $n > 1$ and that the claim holds for all smaller values.

Take $D \in \mathcal{T}(n, A)$ and write $D = M^* \circ \tilde{D} \circ (M^{\text{ad}})^*$ with $M \in \text{Mat}(n, A)$ and $\tilde{D} \in \mathcal{T}(n-1, A[X_n])$. Assume that D has a slice $s \in A[X_1, \dots, X_n]$.

Now there are two cases: $\det(M) \neq 0$ and $\det(M) = 0$.

$\det(M) \neq 0$: Because M is invertible over the quotient field $Q(A)$ of A , the derivation \tilde{D} has a slice as well, viz. $(M^{\text{ad}})^*(s)$. Therefore, by the induction hypothesis, $\tilde{D} \in \mathcal{T}^s(n-1, A[X_n])$ and hence \tilde{D} can in particular be written as

$$\tilde{D} = F \circ a \partial_{X_1} \circ F^{-1},$$

for a certain $a \in A[X_n]^* = A^*$ and a certain automorphism $F \in \text{Aut}_A(A[X_1, \dots, X_n])$ without constant part and with $F(X_n) = X_n$.

Now, Lemma 2.4.12 implies that the last row of M is unimodular. Because A is Hermite, this last row of M can be extended to an invertible

square matrix L over A . Obviously, this matrix L can even be chosen in such a way that it has determinant 1.

Now let \tilde{D} be the derivation

$$\begin{aligned}\hat{D} &:= (L^{-1})^* \circ D \circ L^* \\ &= (ML^{-1})^* \circ \tilde{D} \circ ((ML^{-1})^{\text{ad}})^*.\end{aligned}$$

By construction of L , the last row of ML^{-1} equals $(0 \ \dots \ 0 \ 1)$. Hence, Lemma 2.4.13 implies that \tilde{D} is an element of $\mathcal{T}(n-1, A[X_n])$ and therefore, by the induction hypothesis, even an element of $\mathcal{T}^s(n-1, A[X_n])$. So, D itself is an element of $\mathcal{T}^s(n, A)$.

$\det(M) = 0$: Because $\det(M) = 0$, there exists a matrix $P \in \text{Sl}(Q(A), n)$ such that the last column of MP contains only 0's. Now take

$$\begin{aligned}\hat{D} &:= P^* \circ D \circ (P^{-1})^* \\ &= (MP)^* \circ \tilde{D} \circ ((MP)^{\text{ad}})^*\end{aligned}$$

and note that $(MP)^{\text{ad}}$ is of the form

$$(MP)^{\text{ad}} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ * & \dots & * \end{pmatrix}.$$

So, $\hat{D}(X_i) = 0$ for $i \in \{1, \dots, n-1\}$ and, because the last column of MP is 0, $\hat{D}(X_n) = f(X_1, \dots, X_{n-1})/q$ for some polynomial $f \in A[X_1, \dots, X_{n-1}]$ and some $q \in A \setminus \{0\}$.

Because D has a slice, \hat{D} has a slice over $Q(A)$. Hence f is in fact an element of $A \setminus \{0\}$. Because $D = (P^{-1})^* \circ \hat{D} \circ P^*$ (and because D is defined over A), this means that D is of the form $D = a_1 \partial_{X_1} + \dots + a_n \partial_{X_n}$ for certain $a_1, \dots, a_n \in A$. Because it has a slice, the ideal (a_1, \dots, a_n) of A equals A . So, once again using the assumption that A is Hermite, there is a matrix $R \in \text{Sl}(n, A)$ such that the first column of R^{-1} is (a_1, \dots, a_n) . Then $D = R^* \circ \partial_{X_1} \circ (R^{-1})^*$ and hence $D \in \mathcal{T}^s(n, A)$. \square

Finally, the main result of this section: the Cancellation Problem has an affirmative answer for derivations in $\mathcal{T}(n, A)$. In particular, it has an affirmative answer for derivations in $\mathcal{T}(n, \mathbb{C})$.

Corollary 2.4.14. *Let A be a domain and assume that A and all polynomial rings $A[X_1, \dots, X_k]$, $k \in \mathbb{N}^*$, over A are Hermite. If $D \in \mathcal{T}(n, A)$ has a slice, then its kernel is generated by $n - 1$ elements. \square*

Chapter 3

Reduction Properties

In order to solve questions about polynomial maps, derivations, and other objects in the theory of polynomial maps over arbitrary rings R , it is often convenient to reduce such questions to the case that R is a field. For example, if one has a polynomial map $F: R^n \rightarrow R^n$ over a domain R with $\det JF = 1$ and one wants to know if F is invertible, then Keller's Theorem (Theorem 1.5.4) implies that it is sufficient to check if F is invertible over the quotient field $Q(R)$ of R . So Keller's Theorem reduces the question from a domain to a field.

This chapter studies several techniques for reducing questions over a ring R to questions over “easier” rings. The different techniques are probably best explained by some examples.

Reduced properties

Let R be a ring. The *nilradical* of R , i.e., the ideal of R consisting of all nilpotent elements of R , is denoted by η_R , or simply by η if the ring R is understood. The reduced ring R/η is denoted by \bar{R} .

Consider an R -derivation D on $R[X] := R[X_1, \dots, X_n]$. This derivation induces an \bar{R} -derivation \bar{D} on $\bar{R}[X]$ in the obvious way. Now, Property 3.2.1 says that D is locally nilpotent if and only if \bar{D} is. The property “being locally nilpotent” is therefore called a *reduced property* of a derivation on a polynomial ring.

Prime properties

Given an ideal \mathfrak{a} of R , the derivation D also induces an R/\mathfrak{a} -derivation on $R/\mathfrak{a}[X]$. This derivation is denoted by D/\mathfrak{a} . Property 3.2.3 says that D is locally nilpotent if and only if D/\mathfrak{p} is locally nilpotent for each prime ideal \mathfrak{p} of R , provided that

R is Noetherian. The property “being locally nilpotent” is therefore said to be a *prime property* of a derivation on a polynomial ring over a Noetherian ring.

Local properties

Similarly, given a prime ideal \mathfrak{p} of R , the derivation D induces an $R_{\mathfrak{p}}$ -derivation $D_{\mathfrak{p}}$ on $R_{\mathfrak{p}}[X]$. Property 3.2.2 states that D is locally nilpotent if and only if $D_{\mathfrak{m}}$ is locally nilpotent for every maximal ideal \mathfrak{m} of R . The property “being locally nilpotent” is therefore also said to be a *local property* of a derivation on a polynomial ring.

Residual properties

Consider once again a prime ideal \mathfrak{p} of R . The residue field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ or $Q(R/\mathfrak{p})$ of R at \mathfrak{p} is denoted by $k_{\mathfrak{p}}$, if the ring R is clear from the context. The derivation D induces a $k_{\mathfrak{p}}$ -derivation $\bar{D}_{\mathfrak{p}}$ on $k_{\mathfrak{p}}[X]$. Property 3.2.5 says that D is locally nilpotent if and only if $\bar{D}_{\mathfrak{p}}$ is locally nilpotent for every prime ideal \mathfrak{p} of R , provided that R is Noetherian. The property “being locally nilpotent” is therefore also said to be a *residual property* of a derivation on a polynomial ring over a Noetherian ring.

Quotient properties

Finally, assume that R is a domain. Whenever the ring R is clear from the context, the quotient field of R will simply be denoted by K . The derivation D induces a derivation \tilde{D} on $K[X]$ and once again, by Property 3.2.4, D is locally nilpotent if and only if \tilde{D} is. The property “being locally nilpotent” is therefore said to be a *quotient property* of a derivation on a polynomial ring over a domain.

Along with these properties about locally nilpotent derivations, the most important parts of this chapter are Property 3.7.11, which says that “being a coordinate” is a local property for polynomials over a Hermite domain, and Property 3.6.8, which says that “being a coordinate” is a residual property for polynomials in two variables.

3.1 $\mathfrak{a} = R[X]$

Notation 3.1.1. Let \mathfrak{a} be an ideal of $R[X]$. Then the image of \mathfrak{a} under the natural map $R[X] \rightarrow \bar{R}[X]$ is an ideal of $\bar{R}[X]$. This ideal is denoted by $\bar{\mathfrak{a}}$. Given a prime ideal \mathfrak{p} of $R[X]$, the image of \mathfrak{a} in $R/\mathfrak{p}[X]$ is also an ideal. This ideal is denoted by $\mathfrak{a}/\mathfrak{p}$. The ideal generated by the image of \mathfrak{a} in $R_{\mathfrak{p}}[X]$ is denoted by $\mathfrak{a}_{\mathfrak{p}}$ and the ideal

generated by the image of \mathfrak{a} in $k_{\mathfrak{p}}[X]$ is denoted by $\bar{\mathfrak{a}}_{\mathfrak{p}}$. Finally, if R is a domain, then the ideal of $K[X] = Q(R)[X]$ generated by the image of \mathfrak{a} is denoted by $\tilde{\mathfrak{a}}$.

Reduced Property 3.1.2. *Let $\mathfrak{a} \subseteq R[X]$ be an ideal. Assume that $\bar{\mathfrak{a}} = \bar{R}[X]$. Then $\mathfrak{a} = R[X]$.*

Proof. Since $\bar{\mathfrak{a}} = \bar{R}[X]$, there is an $a \in \mathfrak{a}$ such that $1 \equiv a \pmod{\eta[X]}$. This means that there is an $f \in \eta[X]$ such that $1 = a + f$, or, in other words, $a = 1 - f$. Hence \mathfrak{a} contains a unit. \square

Prime Property 3.1.3. *Let $\mathfrak{a} \subseteq R[X]$ be an ideal. Assume that $\mathfrak{a}/\mathfrak{p} = R/\mathfrak{p}[X]$ for all $\mathfrak{p} \in \text{Spec}(R)$. Then $\mathfrak{a} = R[X]$.*

Proof. Assume that $\mathfrak{a} \neq R[X]$. Then \mathfrak{a} is contained in some maximal ideal of $R[X]$, say \mathfrak{m} . Take $\mathfrak{p} := \mathfrak{m} \cap R$. This is a prime ideal of R .

Since $\mathfrak{a}/\mathfrak{p} = R/\mathfrak{p}[X]$, there is a polynomial $a \in \mathfrak{a}$ such that $a \equiv 1 \pmod{\mathfrak{p}[X]}$. So, $a - 1 \in \mathfrak{p}[X] \subseteq \mathfrak{m}$ and, because $a \in \mathfrak{a} \subseteq \mathfrak{m}$, this means that $1 \in \mathfrak{m}$. So $1 \in \mathfrak{m} \cap R = \mathfrak{p}$, which contradicts the fact that \mathfrak{p} is prime. Hence $\mathfrak{a} = R[X]$. \square

Local Property 3.1.4. *Let $\mathfrak{a} \subseteq R[X]$ be an ideal. Assume that $\mathfrak{a}_{\mathfrak{m}} = R_{\mathfrak{m}}[X]$ for all $\mathfrak{m} \in \text{Max}(R)$. Then $\mathfrak{a} = R[X]$.*

Proof. Since $R_{\mathfrak{m}}[X] = R[X]_{\mathfrak{m}}$, this is just the fact that R -modules that are locally equal are, in fact, equal. See, for instance, [AM69], Proposition 3.9. \square

Counterexample 3.1.5 (Quotient Property). Take $R := \mathbb{Z}$ and consider the ideal $\mathfrak{a} := 2R[X_1]$ of $R[X_1]$. Then $\tilde{\mathfrak{a}} = K[X_1]$, but obviously $\mathfrak{a} \neq R[X_1]$. So “being the unit ideal” is not a quotient property. Assuming that R is a \mathbb{Q} -algebra doesn’t help either: just consider $R := \mathbb{Q}[T]$ and $\mathfrak{a} := T^2R[X_1] \subseteq R[X_1]$.

Even though “being the unit ideal” is not a quotient property, it is still a residual property. The proof proceeds along the same lines as the proof of Property 3.1.3.

Residual Property 3.1.6. *Let $\mathfrak{a} \subseteq R[X]$ be an ideal. Assume that $\bar{\mathfrak{a}}_{\mathfrak{p}} = k_{\mathfrak{p}}[X]$ for all $\mathfrak{p} \in \text{Spec}(R)$. Then $\mathfrak{a} = R[X]$.*

Proof. Assume that $\mathfrak{a} \neq R[X]$. Then \mathfrak{a} is contained in some maximal ideal of $R[X]$, say \mathfrak{m} . Take $\mathfrak{p} := \mathfrak{m} \cap R$. This is a prime ideal of R . Viewing $k_{\mathfrak{p}}$ as $Q(R/\mathfrak{p})$, one can easily see that $1 \in \bar{\mathfrak{a}}_{\mathfrak{p}}$ means that there are an element $r \in R \setminus \mathfrak{p}$, polynomials $g_1, \dots, g_s \in R[X]$, and polynomials $f_1, \dots, f_s \in \mathfrak{a}$ such that

$$r \equiv g_1 f_1 + \dots + g_s f_s \pmod{\mathfrak{p}[X]}.$$

As $\mathfrak{a} \subseteq \mathfrak{m}$ and $\mathfrak{p}[X] \subseteq \mathfrak{m}$ this implies, however, that $r \in \mathfrak{m}$. Because also $r \in R$, this contradicts the fact that $r \notin \mathfrak{p} = \mathfrak{m} \cap R$. Therefore $\mathfrak{a} = R[X]$. \square

3.2 D is locally nilpotent

The following property is essentially Lemma 2.1.15 from [Ess00].

Reduced Property 3.2.1. *Let D be an R -derivation on $R[X]$. Assume that \bar{D} is locally nilpotent (on $\bar{R}[X]$). Then D is locally nilpotent.*

Proof. Let R_0 be the subring of R generated by all coefficients appearing in the polynomials $D(X_1), \dots, D(X_n)$. Then R_0 is Noetherian and D restricts to a derivation D_0 on $R_0[X]$. Note that D is locally nilpotent if and only if D_0 is. Moreover, the nilradical η_0 of R_0 equals $\eta \cap R_0$ and, hence, that \bar{D} is locally nilpotent (on $\bar{R}[X]$) if and only if \bar{D}_0 is (on $\bar{R}_0[X]$). Therefore, it is possible to assume, without loss of generality, that the ring R is Noetherian.

By induction on n it will follow that for every $n \in \mathbb{N}^*$ and every $g \in R[X]$, there exists an $N \in \mathbb{N}$ such that $D^N(g) \in \eta^n[X]$. For $n = 1$, this is just the assumption that \bar{D} is locally nilpotent. Now, assume that the claim holds for n and consider $g \in R[X]$. By the induction hypothesis $D^N(g) \in \eta^n[X]$ for some $N \in \mathbb{N}$, say $D^N(g) = \sum_{\alpha \in A} c_\alpha X^\alpha$, for certain $c_\alpha \in \eta^n$. Since \bar{D} is locally nilpotent, there are $M_\alpha \in \mathbb{N}$ such that $D^{M_\alpha}(X^\alpha) \in \eta[X]$. Taking $M := N + \max_{\alpha \in A} M_\alpha$, it follows that $D^M(g) \in \eta^{n+1}[X]$.

Because R is Noetherian, its nilradical η is finitely generated and hence there is an $e \in \mathbb{N}$ such that $\eta^e = (0)$. Consequently, for every $g \in R[X]$ there is an $N \in \mathbb{N}$ such that $D^N(g) \in \eta^e[X] = (0)$. Therefore, D is locally nilpotent. \square

Local Property 3.2.2. *Let D be an R -derivation on $R[X]$. Assume that $D_{\mathfrak{m}} \in \text{Der}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}[X])$ is locally nilpotent for all $\mathfrak{m} \in \text{Max}(R)$. Then D is locally nilpotent.*

Proof. Form $\mathfrak{a} := \{r \in R \mid \exists N \in \mathbb{N} \forall n \geq N \forall i \in \{1, \dots, n\} [rD^n(X_i) = 0]\}$. Note that this is an ideal of R . The goal is to prove that $1 \in \mathfrak{a}$, since that implies that D is locally nilpotent.

Assume that $1 \notin \mathfrak{a}$. Then \mathfrak{a} is contained in some maximal ideal of R , say \mathfrak{m} . Now $D_{\mathfrak{m}}$ is locally nilpotent and, hence, there is a $K \in \mathbb{N}$ such that for all $k \geq K$ and for all $i \in \{1, \dots, n\}$

$$D_{\mathfrak{m}}^k(X_i) = 0 \quad (\text{in } R_{\mathfrak{m}}[X]).$$

For all $i \in \{1, \dots, n\}$, let $r_i \in R \setminus \mathfrak{m}$ be such that $r_i D^K(X_i) = 0$ (in $R[X]$). Taking $r := r_1 \cdots r_n$ it follows that, for all $i \in \{1, \dots, n\}$, $rD^K(X_i) = 0$. Also,

for all $k \geq K$,

$$\begin{aligned} rD^k(X_i) &= D^{k-K}(rD^K(X_i)) \\ &= D^{k-K}(0) \\ &= 0 \end{aligned}$$

for all $i \in \{1, \dots, n\}$. So $r \in \mathfrak{a}$. Because $\mathfrak{a} \subseteq \mathfrak{m}$, this contradicts the fact that $r(= r_1 \cdots r_n) \notin \mathfrak{m}$. \square

Prime Property 3.2.3. *Assume that R is Noetherian and let D be an R -derivation on $R[X]$. Assume that the derivation $D/\mathfrak{p} \in \text{Der}_{R/\mathfrak{p}}(R/\mathfrak{p}[X])$ is locally nilpotent for all $\mathfrak{p} \in \text{Spec}(R)$. Then D is locally nilpotent.*

Proof. Because R is Noetherian, its nilradical η is a finite intersection of prime ideals, say $\eta = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s$ with $\mathfrak{p}_i \in \text{Spec}(R)$, $i = 1, \dots, s$.

Let $g \in R[X]$. Then, for every $i \in \{1, \dots, s\}$, there exists an $N_i \in \mathbb{N}$ such that $D^{N_i}(g) \in \mathfrak{p}_i[X]$. Taking $N := \max_{i \in \{1, \dots, s\}} N_i$, it follows that $D^N(g) \in \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s = \eta$.

So, \bar{D} is locally nilpotent and hence, by Property 3.2.1, D is locally nilpotent as well. \square

Quotient Property 3.2.4. *Assume that R is a domain and let D be an R -derivation on $R[X]$. Assume that $\tilde{D} \in \text{Der}_K(K[X])$ is locally nilpotent. Then D is locally nilpotent.*

Proof. Obvious. \square

Residual Property 3.2.5. *Assume that R is Noetherian and let D be an R -derivation on $R[X]$. Assume that $\bar{D}_{\mathfrak{p}} \in \text{Der}_{k_{\mathfrak{p}}}(k_{\mathfrak{p}}[X])$ is locally nilpotent for all $\mathfrak{p} \in \text{Spec}(R)$. Then D is locally nilpotent.*

Proof. This is an immediate consequence of Properties 3.2.3 and 3.2.4. \square

In dimension two, the condition that R is Noetherian is not needed for Properties 3.2.3 and 3.2.5. That is caused by the following lemma, which is Theorem 1.3.49 from [Ess00]. It provides a criterium to decide if a derivation is locally nilpotent (over a field of characteristic zero and in dimension two).

Lemma 3.2.6. *Let k be a field of characteristic 0 and take $D \in \text{Der}_k(k[X_1, X_2])$. Assume that $D \neq 0$ and let*

$$d := \max\{\deg_{X_1} D(X_1), \deg_{X_1} D(X_2), \deg_{X_2} D(X_1), \deg_{X_2} D(X_2)\}.$$

(Here, by convention, the degree of 0 is taken to be $-\infty$.) Then D is locally nilpotent if and only if $D^{d+2}(X_1) = D^{d+2}(X_2) = 0$. \square

Prime Property 3.2.7. *Assume that R is a \mathbb{Q} -algebra and let D be an R -derivation on $R[X_1, X_2]$. Assume that for every $\mathfrak{p} \in \text{Spec}(R)$, the derivation $D/\mathfrak{p} \in \text{Der}_{R/\mathfrak{p}}(R/\mathfrak{p}[X_1, X_2])$ is locally nilpotent. Then D is locally nilpotent.*

Proof. Let d be the maximum of $\deg_{X_1} D(X_1)$, $\deg_{X_1} D(X_2)$, $\deg_{X_2} D(X_1)$, and $\deg_{X_2} D(X_2)$.

Consider $\mathfrak{p} \in \text{Spec}(R)$. Note that $Q(R/\mathfrak{p})$ is a field of characteristic 0, since R is a \mathbb{Q} -algebra. Hence, by the previous lemma,

$$(D/\mathfrak{p})^{d+2}(X_1) = (D/\mathfrak{p})^{d+2}(X_2) = 0,$$

or, in other words, $D^{d+2}(X_1) \in \mathfrak{p}[X_1, X_2]$ and $D^{d+2}(X_2) \in \mathfrak{p}[X_1, X_2]$.

Hence $D^{d+2}(X_1) \in \bigcap_{\mathfrak{p} \in \text{Spec}(R)} \mathfrak{p}[X_1, X_2] = \eta[X_1, X_2]$, the nilradical of $R[X_1, X_2]$, and also $D^{d+2}(X_2) \in \eta[X_1, X_2]$. This means that \bar{D} is locally nilpotent and hence, by Lemma 3.2.1, D is locally nilpotent too. \square

Residual Property 3.2.8. *Assume that R is a \mathbb{Q} -algebra and let D be an R -derivation on $R[X_1, X_2]$. Assume that for every $\mathfrak{p} \in \text{Spec}(R)$, the derivation $\bar{D}_{\mathfrak{p}} \in \text{Der}_{k_{\mathfrak{p}}}(k_{\mathfrak{p}}[X_1, X_2])$ is locally nilpotent. Then D is locally nilpotent.*

Proof. This is an immediate consequence of Properties 3.2.7 and 3.2.4. \square

3.3 D has a slice

The following two lemmas are Corollary 2.2 and Lemma 2.3 of [BEM01].

Lemma 3.3.1. *Assume that R is a \mathbb{Q} -algebra. Let D be a locally nilpotent R -derivation on $R[X]$. Then D has a slice if and only if D is surjective.*

Proof. This is an immediate consequence of Theorem 1.1.2. \square

Lemma 3.3.2. *Let D be an R -derivation on $R[X]$. Let \mathfrak{a} and \mathfrak{b} be ideals of R and assume that the derivation D/\mathfrak{a} (on $R/\mathfrak{a}[X]$) has a slice and that the derivation D/\mathfrak{b} (on $R/\mathfrak{b}[X]$) is surjective. Then the derivation $D/\mathfrak{a}\mathfrak{b}$ has a slice (on $R/\mathfrak{a}\mathfrak{b}[X]$). \square*

Lemma 3.3.3. *Assume that R is a \mathbb{Q} -algebra. Let D be a locally nilpotent R -derivation on $R[X]$. Assume that $s \in R[X]$ is such that $D(s) \in R[X]^*$. Then D has a slice.*

Proof Consider the subring R_0 of R generated by the coefficients of $D(X_1), \dots, D(X_n)$, of s , and of $D(s)^{-1}$. Then R_0 is Noetherian and the derivation D restricts to a derivation D_0 on $R_0[X]$, with $s \in R_0[X]$ and $D_0(s) \in R_0[X]^*$. Obviously, if D_0 has a slice, D has a slice as well and hence it is possible to assume, without loss of generality, that R is Noetherian.

By induction on e it will now follow that for every $e \in \mathbb{N}^*$ there is an $s' \in R[X]$ such that $D(s') \equiv 1 \pmod{\eta^e[X]}$.

$e = 1$: Because $D(s) \in R[X]^*$, $D(s) = a + b(X)$ for some $a \in R^*$ and $b(X) \in \eta_{R[X]} = \eta[X]$. Now take $s' := a^{-1}s = 1 + a^{-1}b(X)$. Then $D(s') = 1 + a^{-1}b(X) \equiv 1 \pmod{\eta[X]}$.

$e > 1$: Assume that the claim holds for $e \in \mathbb{N}^*$, say $s' \in R[X]$ with $D(s') = 1 + b(X)$ with $b(X) \in \eta^e[X]$. In Lemma 3.3.2, take $\mathfrak{a} := \eta^e$ and $\mathfrak{b} := \eta$. Now D/\mathfrak{a} and D/\mathfrak{b} have a slice, by the induction hypothesis and by the case $n = 1$ respectively. Since they are locally nilpotent, they are (both) surjective by Lemma 3.3.1. Hence $D/\mathfrak{a}\mathfrak{b} = D/\eta^{e+1}$ has a slice, i.e., there is an $s'' \in R[X]$ such that $D(s'') \equiv 1 \pmod{\eta^{e+1}[X]}$.

Because R is Noetherian, its nilradical η is finitely generated and hence there is an $e \in \mathbb{N}$ such that $\eta^e = (0)$. Now, D/η^e has a slice, but this simply means that D has a slice. \square

Reduced Property 3.3.4. Assume that R is a \mathbb{Q} -algebra and let D be a locally nilpotent R -derivation on $R[X]$. Assume that $\bar{D} \in \text{Der}_{\bar{R}}(\bar{R}[X])$ has a slice. Then D has a slice as well.

Proof Assume that \bar{D} has a slice, say $s \in R[X]$ such that $\bar{D}(\bar{s}) = \bar{1}$. This means that there is an $f \in \eta_{R[X]} = \eta[X]$ such that $D(s) = 1 + f$ and hence $D(s) \in R[X]^*$. By Lemma 3.3.3, D has a slice. \square

Prime Property 3.3.5. Assume that R is a Noetherian \mathbb{Q} -algebra and let D be a locally nilpotent R -derivation on $R[X]$. Assume that $D/\mathfrak{p} \in \text{Der}_{R/\mathfrak{p}}(R/\mathfrak{p}[X])$ has a slice for every $\mathfrak{p} \in \text{Spec}(R)$. Then D has a slice.

Proof. Since R is Noetherian, its nilradical η is a finite intersection of prime ideals of R , say $\eta = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_m$. By repeated application of Lemma 3.3.2 (also using Lemma 3.3.1), it follows that D has a slice modulo $\mathfrak{p}_1 \dots \mathfrak{p}_m$. In particular, D has a slice modulo η and then Property 3.3.4 implies that D has a slice. \square

Lemma 3.3.6. For every maximal ideal \mathfrak{m} of R , let $s_{\mathfrak{m}}$ be an element of $R \setminus \mathfrak{m}$. Form $\mathfrak{a} := \sum_{\mathfrak{m} \in \text{Max}(R)} R s_{\mathfrak{m}}$. Then $\mathfrak{a} = R$.

Proof Obviously \mathfrak{a} is an ideal of R . Suppose that $\mathfrak{a} \neq R$. Then \mathfrak{a} is contained in some maximal ideal \mathfrak{m}_0 of R . Then $s_{\mathfrak{m}_0} \in \mathfrak{a} \subseteq \mathfrak{m}_0$, contradicting the assumption that $s_{\mathfrak{m}_0} \notin \mathfrak{m}_0$. Therefore $\mathfrak{a} = R$. \square

Local Property 3.3.7. *Let D be an R -derivation of $R[X]$. Assume that $D_{\mathfrak{m}} \in \text{Der}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}[X])$ has a slice for every $\mathfrak{m} \in \text{Max}(R)$. Then D has a slice.*

Proof For every $\mathfrak{m} \in \text{Max}(R)$, let $f_{\mathfrak{m}} \in R[X]$ and $s_{\mathfrak{m}} \in R \setminus \mathfrak{m}$ be such that $D_{\mathfrak{m}}(f_{\mathfrak{m}}/s_{\mathfrak{m}}) = 1 \in R_{\mathfrak{m}}[X]$. This means that $D_{\mathfrak{m}}(f_{\mathfrak{m}}) = s_{\mathfrak{m}} \in R_{\mathfrak{m}}[X]$ and hence

$$u_{\mathfrak{m}}(D(f_{\mathfrak{m}}) - s_{\mathfrak{m}}) = 0 \in R[X]$$

for a certain $u_{\mathfrak{m}} \in R \setminus \mathfrak{m}$. So $D(u_{\mathfrak{m}}f_{\mathfrak{m}}) = u_{\mathfrak{m}}s_{\mathfrak{m}}$.

Note that $u_{\mathfrak{m}}s_{\mathfrak{m}} \in R \setminus \mathfrak{m}$. According to Lemma 3.3.6, $\sum_{\mathfrak{m} \in \text{Max}(R)} Ru_{\mathfrak{m}}s_{\mathfrak{m}} = R$, so there is a finite subset $M \subseteq \text{Max}(R)$ and elements $a_{\mathfrak{m}} \in R$, for $\mathfrak{m} \in M$, such that

$$1 = \sum_{\mathfrak{m} \in M} a_{\mathfrak{m}} u_{\mathfrak{m}} s_{\mathfrak{m}}.$$

Now take $f := \sum_{\mathfrak{m} \in M} a_{\mathfrak{m}} u_{\mathfrak{m}} s_{\mathfrak{m}} f_{\mathfrak{m}}$. Then $D(f) = 1$. \square

Counterexample 3.3.8 (Residual Property). Consider the ring $R := \mathbb{C}[T]$ and the derivation

$$D := (1 + Z^2)\partial_X + Z\partial_Y + T\partial_Z$$

on $R[X, Y, Z]$. Then D has a slice over every residue field $k_{\mathfrak{p}}$, $\mathfrak{p} \in \text{Spec}(R)$, but D itself has no slice. This was shown by Deveney, Finston, and Gehrke in [DFG94], but will also follow from Theorem 5.3.5 (see Example 5.3.6). Furthermore, D is triangular and hence locally nilpotent. So “having a slice” is not a residual property of a locally nilpotent derivation.

Counterexample 3.3.9 (Quotient Property). Of course, “having a slice” isn’t a quotient property of a derivation either. Just look at the derivation $D := T\partial_X$ over the same ring $R := \mathbb{C}[T]$. Over $K = \mathbb{C}(T)$ it has a slice, viz., $T^{-1}X$, but over R it hasn’t.

3.4 $R[X]^D$ is finitely generated over R

Counterexample 3.4.1 (Reduced and Prime Property). Consider the R defined by $R := \mathbb{C}[T]/(T^2)$, say $\epsilon = \bar{T}$. Take $D := \epsilon\partial_{X_1}$ on $R[X_1]$. Then, on the one hand,

$$R[X_1]^D = R[\epsilon X_1, \epsilon X_1^2, \dots]$$

and one cannot find a finite number of generators for this kernel (just look at ϵX_1^m , where m is larger than the degree of all generators). On the other hand, $\bar{R}[X_1]^{\bar{D}} = \bar{R}[X_1]$, so this is finitely generated. Therefore “having a finitely generated kernel” is not a reduced property of a locally nilpotent derivation.

Since $\eta = (\epsilon)$ is also the only prime ideal of R , it also follows that “having a finitely generated kernel” is not a prime property of a locally nilpotent derivation.

Counterexample 3.4.2 (Quotient and Residual Property). Consider a field k of characteristic zero and consider the locally nilpotent derivation

$$X^3\partial_S + S\partial_T + T\partial_U + X^2\partial_V$$

on $k[S, T, U, V, X]$. As was already remarked in Section 1.3, Daigle and Freudenburg have proven that the kernel of this derivation is not a finitely generated k -algebra. They have also remarked that one can change this derivation slightly by conjugation with the polynomial automorphism over k sending S to $S - XV$ and the other variables to themselves. Then one gets the derivation

$$D := (S + XV)\partial_T + T\partial_U + X^2\partial_V.$$

This derivation can be considered as an R -derivation on $R[T, U, V]$, where $R := k[S, X]$. By Daigle and Freudenburg’s result, the kernel of this derivation is not finitely generated over k and therefore it isn’t finitely generated over R either. Miyanishi’s result (Theorem 1.3.5), however, implies that for every prime ideal \mathfrak{p} of R , the kernel of $\bar{D}_{\mathfrak{p}}$ on $k_{\mathfrak{p}}[T, U, V]$ is a finitely generated $k_{\mathfrak{p}}$ -algebra. So “having a finitely generated kernel” is not a residual property of a locally nilpotent derivation.

The same result implies that the derivation \tilde{D} on $K[T, U, V]$ has a finitely generated kernel over K . So “having a finitely generated kernel” is also not a quotient property of a locally nilpotent derivation.

3.5 F is invertible

Notation 3.5.1. Let $F: R^n \rightarrow R^n$ be a polynomial map. The same notations as before (for derivations, for instance) will be used: $\bar{F}: \bar{R}^n \rightarrow \bar{R}^n$, $F_{\mathfrak{p}}: R_{\mathfrak{p}}^n \rightarrow R_{\mathfrak{p}}^n$, $F/\mathfrak{p}: R/\mathfrak{p}^n \rightarrow R/\mathfrak{p}^n$, $\tilde{F}: K^n \rightarrow K^n$, and $\bar{F}_{\mathfrak{p}}: k_{\mathfrak{p}}^n \rightarrow k_{\mathfrak{p}}^n$ all denote the obvious induced polynomial maps.

The properties in this section are all well-known. See, for instance, Remark 1.1 of the very influential paper [BCW82] of Bass, Connell, and Wright.

Reduced Property 3.5.2. Let $F: R^n \rightarrow R^n$ be a polynomial map. Assume that \bar{F} is invertible. Then F is invertible. \square

Proof Because \bar{F} is invertible, $\overline{\det JF} = \det J\bar{F} \in \bar{R}[X]^*$ and therefore it follows that $\det JF \in R[X]^*$. Without loss of generality, assume that $F(0) = 0$.

Now, F is invertible if and only if all R -derivations $\partial/\partial F_i$ are locally nilpotent. This follows, for instance, from Proposition 1.5.6. Now use Property 3.2.1. \square

Local Property 3.5.3. Let $F: R^n \rightarrow R^n$ be a polynomial map. Assume that for every $\mathfrak{m} \in \text{Max}(R)$ the polynomial map $F_{\mathfrak{m}}: R_{\mathfrak{m}}^n \rightarrow R_{\mathfrak{m}}^n$ is invertible. Then F is invertible.

Proof Such a polynomial map F can also be seen as a homomorphism $F^*: R[X] \rightarrow R[X]$ of R -algebras. By Proposition 3.9 of [AM69], invertibility is a local property. This means that F^* is invertible if and only if $F_{\mathfrak{m}}^*: R[X]_{\mathfrak{m}} \rightarrow R[X]_{\mathfrak{m}}$ is invertible for all maximal ideals \mathfrak{m} of R . Now note that $R[X]_{\mathfrak{m}} = R_{\mathfrak{m}}[X]$. \square

Second proof: The assumption states that, for every $\mathfrak{m} \in \text{Max}(R)$,

$$R_{\mathfrak{m}}[F] = R_{\mathfrak{m}}[X].$$

(Note that $R_{\mathfrak{m}}[F] = R[F]_{\mathfrak{m}}$ and that $R_{\mathfrak{m}}[X] = R[X]_{\mathfrak{m}}$). Furthermore, $R[F] \subseteq R[X]$ and invertibility of F is equivalent to $R[F] = R[X]$.

Take $f \in R[X]$ and define $\mathfrak{a} := \{r \in R \mid rf \in R[F]\}$. This is an ideal of R . Suppose that $1 \notin \mathfrak{a}$. Then $\mathfrak{a} \subseteq \mathfrak{m}_0$ for some maximal ideal \mathfrak{m}_0 of R . Now $R_{\mathfrak{m}_0}[F] = R_{\mathfrak{m}_0}[X]$ means that $sf \in R[F]$ for some $s \in R \setminus \mathfrak{m}_0$. Then, however, $s \in \mathfrak{a} \subseteq \mathfrak{m}_0$, which gives a contradiction. Hence $1 \in \mathfrak{a}$ and therefore $f \in R[F]$. So $R[F] = R[X]$ and F is invertible. \square

Prime Property 3.5.4. Let $F: R^n \rightarrow R^n$ be a polynomial map. Assume that $F/\mathfrak{p}: (R/\mathfrak{p})^n \rightarrow (R/\mathfrak{p})^n$ is invertible, for all $\mathfrak{p} \in \text{Spec}(R)$. Then F is invertible.

Proof. It is possible to assume, without loss of generality, that $F(0) = 0$. Consider the ideal $\mathfrak{a} := (\det(JF))$ of $R[X]$. Because F/\mathfrak{p} is invertible, $\mathfrak{a}/\mathfrak{p} (= (\det(J(F/\mathfrak{p})))) = R/\mathfrak{p}[X]$, for all $\mathfrak{p} \in \text{Spec}(R)$. Hence, by Property 3.1.3, $\det(JF)$ is a unit in $R[X]$.

Now let $G \in R[[X]]^n$ be the formal inverse of F . Note that $G/\mathfrak{p}: (R/\mathfrak{p})^n \rightarrow (R/\mathfrak{p})^n$ is the (polynomial) inverse of F/\mathfrak{p} , for all $\mathfrak{p} \in \text{Spec}(R)$. Because each R/\mathfrak{p} is a domain, $\deg G/\mathfrak{p} \leq (\deg F/\mathfrak{p})^{n-1} \leq (\deg F)^{n-1}$ by Corollary 1.4 of [BCW82].

So the coefficients of all terms of G of degree greater than $(\deg F)^{n-1}$ turn out to be elements of all prime ideals of R and hence of the nilradical η of R . So \bar{G} is a polynomial map and therefore \bar{F} is invertible. By Proposition 3.5.2, F is invertible too. \square

Quotient Property 3.5.5. Assume that R is a domain and let $F: R^n \rightarrow R^n$ be a polynomial map with $\det(JF) \in R^*$. Assume that $\bar{F}: K^n \rightarrow K^n$ is invertible. Then F is invertible.

Proof. This is just a special case of Keller's Theorem (Theorem 1.5.4). \square

Counterexample 3.5.6 (Quotient Property). Of course, if one omits the condition that $\det(JF) \in R[X]^*$, the above proposition is false. Just take $R := \mathbb{C}[T]$ and $F := (TX_1): R^1 \rightarrow R^1$.

Residual Property 3.5.7. Let $F: R^n \rightarrow R^n$ be a polynomial map. Assume that $\bar{F}_{\mathfrak{p}}: k_{\mathfrak{p}}^n \rightarrow k_{\mathfrak{p}}^n$ is invertible, for all $\mathfrak{p} \in \text{Spec}(R)$. Then F is invertible.

Proof. Just as in the proof of Property 3.5.4 consider the ideal $\mathfrak{a} := (\det(JF))$ of $R[X]$. Because $\bar{F}_{\mathfrak{p}}$ is invertible, $\bar{\mathfrak{a}}_{\mathfrak{p}} (= (\det(J\bar{F}_{\mathfrak{p}}))) = k_{\mathfrak{p}}[X]$, for all $\mathfrak{p} \in \text{Spec}(R)$. Hence, by Property 3.1.6, $\det(JF)$ is a unit in $R[X]$.

Now, for every $\mathfrak{p} \in \text{Spec}(R)$, consider $k_{\mathfrak{p}}$ as $Q(R/\mathfrak{p})$. Since $\det(JF) \in R[X]^*$, it follows that $\det(JF/\mathfrak{p})$ is a unit in R/\mathfrak{p} and hence Property 3.5.5 says that F/\mathfrak{p} is invertible. By Property 3.5.4, F is invertible too. \square

3.6 f is a coordinate

Reduced Property 3.6.1. Let $f \in R[X]$. Assume that \bar{f} is a coordinate over \bar{R} . Then f is a coordinate.

Proof. Take $f_2, \dots, f_n \in R[X]$ such that $(\bar{f}, \bar{f}_2, \dots, \bar{f}_n)$ is an invertible polynomial map over \bar{R} . By Property 3.5.2, (f, f_2, \dots, f_n) is then invertible too. So f is a coordinate. \square

Lemma 3.6.2. *Let R be a ring and $D \in \text{Der}_R(R[X])$. Let \mathfrak{p} be a prime ideal of R . Then $(R[X]^D)_{\mathfrak{p}} = (R_{\mathfrak{p}}[X])^{D_{\mathfrak{p}}}$.*

Proof. \subseteq : Take an element of $(R[X]^D)_{\mathfrak{p}}$, say f/s with $f \in R[X]^D$ and $s \in R \setminus \mathfrak{p}$. Then $D(f) = 0$ in $R[X]$, so $D_{\mathfrak{p}}(f) = 0$ in $R_{\mathfrak{p}}[X]$ as well. Therefore, $D_{\mathfrak{p}}(f/s) = D_{\mathfrak{p}}(f)/s = 0$ in $R_{\mathfrak{p}}[X]$.

\supseteq : Take an element of $(R_{\mathfrak{p}}[X])^{D_{\mathfrak{p}}}$, say f/s with $f \in R[X]$, $s \in R \setminus \mathfrak{p}$. Then $D_{\mathfrak{p}}(f/s) = 0$, so $D_{\mathfrak{p}}(f) = 0$ as well. This means that

$$tD(f) = 0 \in R[X]$$

for some $t \in R \setminus \mathfrak{p}$. Hence $f/s = (tf)/(ts)$ with $D(tf) = 0$ and $ts \in R \setminus \mathfrak{p}$. \square

Counterexample 3.6.3 (Local and Residual Property). Consider the derivation $D := a\partial_{X_1} + b\partial_{X_2} + c\partial_{X_3}$ on the polynomial ring $R[X_1, X_2, X_3]$, where R is the ring $\mathbb{R}[a, b, c]/(a^2 + b^2 + c^2 - 1)$. This derivation is locally nilpotent and has a (linear) slice $s := aX_1 + bX_2 + cX_3$. Localising in a prime ideal \mathfrak{p} of R , $D_{\mathfrak{p}}$ is locally nilpotent and has a slice s as well. Since every local ring is Hermite (see, e.g., [Lam78], Corollary 1.8), Proposition 2.2.6 implies that $(R_{\mathfrak{p}}[X])^{D_{\mathfrak{p}}}$ is generated by $n - 1$ elements. It also follows from Proposition 2.2.6, however, that $R[X]^D$ is not generated, as an R -algebra, by $n - 1$ elements. (See also Example 2.2.8). Therefore, s is not a coordinate. This means that “being a coordinate” is not a local property.

As a consequence, “being a coordinate” is also not a residual property, for if a polynomial f is a coordinate in $R_{\mathfrak{p}}[X]$, it is a coordinate in $k_{\mathfrak{p}}[X]$ as well.

Also note that, by Lemma 3.6.2, $R[X]^D$ is an example of an R -algebra that is locally generated by $n - 1$ elements, but for which one globally needs at least n generators.

Counterexample 3.6.4 (Quotient Property). A very easy example shows that the property of “being a coordinate” is not a quotient property of a polynomial. Just consider $f := TX_1 \in R[X_1, \dots, X_n]$ over $R := \mathbb{C}[T]$.

From now on, this section studies the case of two variables.

Lemma 3.6.5. *Assume that R is a \mathbb{Q} -algebra. Take $f \in R[X_1, X_2]$ and let D be the R -derivation $f_{X_2}\partial_{X_1} - f_{X_1}\partial_{X_2}$ on $R[X_1, X_2]$. Assume that f is a coordinate in $R[X_1, X_2]$. Then D is locally nilpotent and has a slice.*

Proof Let $g \in R[X_1, X_2]$ be a polynomial such that (f, g) is an invertible polynomial map over R . Then (\bar{f}, \bar{g}) is an invertible polynomial map over \bar{R} and hence $\bar{R}[X_1, X_2] = \bar{R}[\bar{f}, \bar{g}]$. Now note that

$$\bar{D}(\bar{g}) = \det J(\bar{f}, \bar{g}) \in \bar{R}[X_1, X_2]^* = \bar{R}^*$$

and so $\bar{D}^2(\bar{g}) = 0$. Also $\bar{D}(\bar{f}) = 0$ and therefore \bar{D} is locally nilpotent. By Lemma 3.2.1, D is locally nilpotent too.

So in view of Theorem 1.3.4, the only thing left to show is that 1 is an element of the ideal generated by $D(X_1)$ and $D(X_2)$. Now $\det J(f, g) \in R[X_1, X_2]^*$. So $g_{X_1}f_{X_2} - g_{X_2}f_{X_1}$ is invertible. Hence the ideal generated by $D(X_1) = f_{X_2}$ and $D(X_2) = -f_{X_1}$ contains an invertible element and consequently contains 1. Therefore, as observed, D has a slice. \square

Proposition 3.6.6. *Assume that R is a \mathbb{Q} -algebra. Take $f \in R[X_1, X_2]$ and let D be the derivation $f_{X_2}\partial_{X_1} - f_{X_1}\partial_{X_2} \in \text{Der}_R(R[X_1, X_2])$. Then the following three statements are equivalent:*

1. D is locally nilpotent and $(f_{X_1}, f_{X_2}) = R[X_1, X_2]$;
2. D is locally nilpotent, has a slice, and $R[X_1, X_2]^D = R[f]$;
3. f is a coordinate.

Proof The equivalence of 1 and 2 is Theorem 1.3.4 the equivalence of 2 and 3 follows from Theorem 1.1.2 and the preceding lemma. \square

Prime Property 3.6.7. *Assume that R is a \mathbb{Q} -algebra and let $f \in R[X_1, X_2]$. Assume that f is a coordinate in $R/\mathfrak{p}[X_1, X_2]$ for all $\mathfrak{p} \in \text{Spec}(R)$. Then f is a coordinate.*

Proof This follows from Proposition 3.6.6 and Properties 3.1.3 and 3.2.7. \square

Now it is also possible to prove that “being a coordinate” is a residual property for arbitrary \mathbb{Q} -algebras in two variables. This slightly extends a result from Bhatwadekar and Dutta in [BD93], which proves the following statement for a Noetherian ring which either contains \mathbb{Q} or for which R/η is seminormal.

Residual Property 3.6.8. *Assume that R is a \mathbb{Q} -algebra and let $f \in R[X_1, X_2]$. Then the following two statements are equivalent:*

1. f is a coordinate over R in $R[X_1, X_2]$;
2. for every $\mathfrak{p} \in \text{Spec}(R)$, $\bar{f}_{\mathfrak{p}}$ is a coordinate over $k_{\mathfrak{p}}$ in $k_{\mathfrak{p}}[X_1, X_2]$.

Proof. This is now an easy consequence of the equivalence $1 \Leftrightarrow 3$ from the previous proposition and from Properties 3.1.6 and 3.2.8. \square

Example 3.6.9. The condition that R is a \mathbb{Q} -algebra cannot simply be dropped in this proposition. For Bhatwadekar and Dutta have constructed the following example in [BD93]. Take $R := \mathbb{Z}_{2\mathbb{Z}}[2\sqrt{2}]$ and take

$$f := X_1 - 2X_2(\sqrt{2}X_1 - X_2^2) + \sqrt{2}(\sqrt{2}X_1 - X_2^2)^2 - \sqrt{2}(X_2 - \sqrt{2}(\sqrt{2}X_1 - X_2^2))^4.$$

Then $\bar{f}_{\mathfrak{p}}$ is a coordinate over $k_{\mathfrak{p}}$, for every prime ideal \mathfrak{p} of R (there are only two), but f itself is not a coordinate over R .

Local Property 3.6.10. Assume that R is a \mathbb{Q} -algebra and let $f \in R[X_1, X_2]$. Then the following two statements are equivalent:

1. f is a coordinate over R in $R[X_1, X_2]$;
2. for every $\mathfrak{p} \in \text{Spec}(R)$, $\bar{f}_{\mathfrak{p}}$ is a coordinate over $R_{\mathfrak{p}}$ in $R_{\mathfrak{p}}[X_1, X_2]$.

Proof. This follows immediately from Property 3.6.8: if a polynomial is a coordinate locally, then it certainly is a coordinate residually. \square

Example 3.6.3 shows that “being a coordinate” is in general not a local property. However, if one additionally assumes that the ring R is a Hermite domain, then it is. The reasoning is rather lengthy compared to the other properties treated in this chapter. Therefore, it is treated separately in the next section.

3.7 Local coordinates

Let R be a domain, $n \in \mathbb{N}^*$, and $R[X] := R[X_1, \dots, X_n]$ the polynomial ring in n variables over R . This section shows that a polynomial in $R[X]$ is a coordinate if and only if it is a coordinate when considered as an element of $R_{\mathfrak{m}}[X]$, for all maximal ideals \mathfrak{m} of R , provided that R is Hermite, and similarly for partial coordinate systems.

The ideas presented in this section can in fact already be found in [Qui76]. The abstract notion of Quillen Induction is essentially taken from [BCW77] and the results from that paper can also be used to derive the main result of this section. Together with Section 4.5, this section forms the main part of [DER00].

Definition 3.7.1. Define $\text{Loc}(R) := \{R_r \mid r \in R \setminus \{0\}\}$.

Proposition 3.7.2 (Quillen Induction). *Let $P \subseteq \text{Loc}(R)$. Write $P(L)$ instead of $L \in P$ for $L \in \text{Loc}(R)$. In that case, L is said to have property P . Assume that*

- (a) *for all $\mathfrak{m} \in \text{Max}(R)$: there exists an $r \in R \setminus \mathfrak{m}$ such that $P(R_r)$;*
- (b) *for all $r, s, t \in R \setminus \{0\}$: if $rR_t + sR_t = R_t$, $P(R_r)$, and $P(R_s)$, then $P(R_t)$.*

Then $P(L)$ for all $L \in \text{Loc}(R)$. In particular $P(R)$.

Proof Let S be the collection of all $r \in R \setminus \{0\}$ such that $P(R_r)$ together with 0. This is an ideal of R . It is not empty because $0 \in S$, closed under addition because of (b) (for $r, s \in S$, take $t := r + s$), and closed under multiplication with elements of R also because of (b) (for $\tilde{r} \in R$ and $\tilde{s} \in S$, take $r := \tilde{s}$, $s := \tilde{s}$, and $t := \tilde{r}\tilde{s}$).

Suppose that $S \neq R$. Then S is contained in some maximal ideal of R , say \mathfrak{m} . By (a) there is an $r \in R \setminus \mathfrak{m}$ such that $P(R_r)$. But then $r \in S \subseteq \mathfrak{m}$, which contradicts $r \notin \mathfrak{m}$. So $S = R$ and therefore $P(L)$ for all $L \in \text{Loc}(R)$. \square

Definition 3.7.3. A polynomial map $H: R^n \rightarrow R^n$ is called *nice* if it is of the form $H = (X_1 + \text{h.o.t.}, \dots, X_n + \text{h.o.t.})$. Here h.o.t. stands for higher order terms, i.e., terms of degree 2 or greater. A coordinate $h \in R[X]$ is called *nice* if there is a nice polynomial automorphism $H: R^n \rightarrow R^n$ with h as its first component. Similarly, a partial coordinate system $(h_1, \dots, h_k) \in R[X]^k$ is called *nice* if there is a nice polynomial automorphism $H: R^n \rightarrow R^n$ which has (h_1, \dots, h_k) as its first k components.

Lemma 3.7.4. *A partial coordinate system $(h_1, \dots, h_k) \in R[X]^k$ is nice if and only if it is of the form $(X_1 + \text{h.o.t.}, \dots, X_k + \text{h.o.t.})$. In particular, a coordinate $h \in R[X]$ is nice if and only if it is of the form $X_1 + \text{h.o.t.}$.*

Proof By linear algebra, looking at the linear part of a polynomial automorphism without constant parts with h_1, \dots, h_k as its first k components. \square

Definition 3.7.5. Let $H: R^n \rightarrow R^n$ be a nice polynomial map. Then the polynomial map ${}^TH: R[T]^n \rightarrow R[T]^n$ over $R[T]$ is defined by

$${}^TH := T^{-1}H[X_1 := TX_1, \dots, X_n := TX_n].$$

(This is defined over $R[T]$ and not just over $R[T, T^{-1}]$ because H is nice.) If $r \in R$, then ${}^TH[T := r] \in \text{End}_R(R[X])$ is denoted by rH .

One can easily see that $(\det JH)[X := TX] = \det J{}^TH$ and that H is invertible if and only if TH is. Even better, if $r \in R \setminus \{0\}$, then $\det J{}^rH \in R^*$ if and only if $\det JH \in R^*$ and rH is invertible if and only if H is.

The map TH is called the *clearing map* because of the following: if K is the quotient field of R and $H: K^n \rightarrow K^n$ is a polynomial map over K of the form $H = X + \text{h.o.t.}$, then there is an $r \in R \setminus \{0\}$ such that rH is in fact defined over R . So, the denominators of H are cleared. See Chapter 1 of [Ess00].

Lemma 3.7.6. *Let $r, s \in R \setminus \{0\}$ be such that $rR + sR = R$ and let $H: R_{rs}^n \rightarrow R_{rs}^n$ be a nice polynomial automorphism over R_{rs} . Then there are nice polynomial automorphisms $H_1: R_r^n \rightarrow R_r^n$ and $H_2: R_s^n \rightarrow R_s^n$ over R_r and over R_s , respectively, such that $H = H_1H_2$.*

Proof. Note that

$${}^TH = H_{(1)} + TH_{(2)} + T^2H_{(3)} + \cdots + T^{d-1}H_{(d)}$$

where each $H_{(i)}$ is the homogeneous part of degree i of H and d is the degree of H . Hence

$$\begin{aligned} {}^{1-T}H &= H_{(1)} + (1-T)H_{(2)} + (1-T)^2H_{(3)} + \cdots + (1-T)^{d-1}H_{(d)} \\ &= H_{(1)} + H_{(2)} + H_{(3)} + \cdots + H_{(d)} + T(\text{h.o.t.}) \\ &= H + T(\text{h.o.t.}), \end{aligned}$$

where, as before, h.o.t. stands for some terms of X -degree at least two. As a consequence

$$\begin{aligned} H^{-1} \circ {}^{1-T}H &= H^{-1} \circ (H + T(\text{h.o.t.})) \\ &= X + T(\text{h.o.t.}). \end{aligned}$$

Now let $k \in \mathbb{N}$ be sufficiently large. From $rR + sR = R$ it follows that $r^kR + s^kR = R$. Take $v, w \in R$ with $r^kv + s^kw = 1$. If k is sufficiently large, then ${}^{s^kw}H$ and ${}^{s^kw}(H^{-1})$ can be seen as polynomial maps $R_r^n \rightarrow R_r^n$ over R_r . They are also each others inverse and hence they are in fact polynomial automorphisms over R_r .

Take $H_1 := {}^{s^kw}H$ and compute $H^{-1}H_1$. This gives

$$\begin{aligned} H^{-1}H_1 &= H^{-1} \circ ({}^TH [T := s^kw]) \\ &= H^{-1} \circ ({}^{1-T}H [T := r^kv]) \\ &= (H^{-1} \circ {}^{1-T}H) [T := r^kv] \\ &= (X + T(\text{h.o.t.})) [T := r^kv] \\ &= X + r^kv(\text{h.o.t.}) \end{aligned}$$

and similarly

$$H_1^{-1}H = X + r^k v(\text{h.o.t.}).$$

For k sufficiently large, $H_2 := H_1^{-1}H$ and its inverse apparently are polynomial automorphisms over R_s . So now $H = H_1 H_2$ with H_1 and H_2 are both of the required form. \square

Lemma 3.7.7. *Let $r, s \in R$ be such that $rR + sR = R$. Take $t \in R_{rs}$ such that $t \in R_r \cap R_s$. Then $t \in R$.*

Proof. Write $t = v/r^k = w/s^l$ with $v, w \in R$ and $k, l \in \mathbb{N}$. Because $rR + sR = R$, also $r^k R + s^l R = R$. Write $r^k x + s^l y = 1$ for some $x, y \in R$. Then $t = (r^k x + s^l y)t = vx + wy \in R$. \square

Lemma 3.7.8 (Patching Lemma). *Let $r, s \in R$ with $rR + sR = R$. Let $k \in \{1, \dots, n\}$ and let $h_1, \dots, h_k \in R[X]$ be polynomials of the form $h_i = X_i + \text{h.o.t.}$. Assume that there is a nice polynomial automorphism $F: R_r^n \rightarrow R_r^n$ over R_r with first k components equal to h_1, \dots, h_k and that there is a nice polynomial automorphism $G: R_s^n \rightarrow R_s^n$ over R_s with first k components also equal to h_1, \dots, h_k . Then there is a nice polynomial automorphism $H: R^n \rightarrow R^n$ over R with first k components equal to h_1, \dots, h_k .*

Proof. Consider the polynomial map $F^{-1}G: R_{rs}^n \rightarrow R_{rs}^n$. The first k components of this polynomial map are (X_1, \dots, X_k) , say

$$GF^{-1} = (X_1, \dots, X_k, p_{k+1}, \dots, p_n).$$

Let $P: R[X_1, \dots, X_k]^{n-k} \rightarrow R[X_1, \dots, X_k]^{n-k}$ be the polynomial map given by (p_{k+1}, \dots, p_n) (in the variables X_{k+1}, \dots, X_n). Applying Lemma 3.7.6 to the ring $R[X_1, \dots, X_k]$, this polynomial map can be written as $P = P_1 P_2$ with P_1 a nice polynomial automorphism over $R[X_1, \dots, X_k]_r = R_r[X_1, \dots, X_k]$ of $R_r[X_1, \dots, X_k]^{n-k}$ and P_2 a nice polynomial automorphism over $R[X_1, \dots, X_k]_s = R_s[X_1, \dots, X_k]$ of $R_s[X_1, \dots, X_k]^{n-k}$ (in the variables X_{k+1}, \dots, X_n).

Now take $H_1 := (X_1, \dots, X_k, P_1)$ and $H_2 := (X_1, \dots, X_k, P_2)$. Then H_1 and H_2 are nice polynomial automorphism over R_r and R_s , respectively. Note that $H_1 H_2 = GF^{-1}$. Hence $H := H_1 F = H_2^{-1} G$ is a nice polynomial automorphism (over R_{rs} , a priori) whose first k components equal h_1, \dots, h_k . It is defined over R_r (because $H = H_1 F$ and F and H_1 are defined over R_r) and it is defined over R_s (because $H = H_2^{-1} G$ and G and H_2 are defined over R_s). Hence, applying Lemma 3.7.7 to every one of its coefficients, it is in fact defined over R . \square

Remark 3.7.9. In terms of R_{rs} -automorphisms of the algebra $R_{rs}[X_1, \dots, X_n]$, the above transfer from $F^{-1}G$ to P (and from P_i to H_i) can be described much more elegantly. $(F^{-1}G)^*$ is an element of $\text{Aut}_{R_{rs}}(R_{rs}[X])$ that leaves X_1, \dots, X_k fixed. Hence it can also be considered as an element of $\text{Aut}_{R_{rs}[X_1, \dots, X_k]}(R_{rs}[X])$. This element is P^* .

Theorem 3.7.10. *Let $k \in \{1, \dots, n\}$ and let $h_1, \dots, h_k \in R[X]$ be polynomials of the form $h_i = X_i + \text{h.o.t.}$. Assume that for every maximal ideal \mathfrak{m} of R , (h_1, \dots, h_k) is a nice partial coordinate system when considered as an element of $R_{\mathfrak{m}}[X]^k$. Then (h_1, \dots, h_k) is a nice partial coordinate system.*

Proof Let $P \subseteq \text{Loc}(R)$ be the collection of all R_r , $r \in R \setminus \{0\}$, such that (h_1, \dots, h_k) is a nice partial coordinate system over R_r . Now check the two conditions for Quillen Induction.

- (a) Let \mathfrak{m} be a maximal ideal of R . It is assumed that (h_1, \dots, h_k) is a nice partial coordinate system over $R_{\mathfrak{m}}$. Using Lemma 3.7.4, choose a polynomial automorphism $F: R_{\mathfrak{m}}^n \rightarrow R_{\mathfrak{m}}^n$ over $R_{\mathfrak{m}}$ with first k components equal to h_1, \dots, h_k . There are only finitely many elements of R appearing in the denominator of a coefficient of a component of F and its inverse. Denote the product of these denominators by r . None of these denominators is an element of \mathfrak{m} and, because \mathfrak{m} is prime, r is not an element of \mathfrak{m} either. Furthermore, obviously, $P(R_r)$.
- (b) Let $r, s, t \in R \setminus \{0\}$ be such that $rR_t + sR_t = R_t$ and assume $P(R_r)$ and $P(R_s)$. Then $P(R_t)$ follows by applying the Patching Lemma (Lemma 3.7.8) to the ring R_t .

So, using Quillen Induction (Proposition 3.7.2), it follows that $P(R)$ holds, which means that (h_1, \dots, h_k) is a nice partial coordinate system over R . \square

Local Property 3.7.11. *Assume that R is Hermite. Let $k \in \{1, \dots, n\}$ and let h_1, \dots, h_k be polynomials in $R[X]$. Assume that (h_1, \dots, h_k) is a partial coordinate system when considered as an element of $R_{\mathfrak{m}}[X]^k$, for every maximal ideal \mathfrak{m} of R . Then (h_1, \dots, h_k) is a partial coordinate system.*

Proof. First of all note that it is possible to assume that the h_i have no constant part. Write $h_i = r_{i1}X_1 + \dots + r_{in}X_n + \text{h.o.t.}$ for all i , with $r_{ij} \in R$.

Consider a maximal ideal \mathfrak{m} of R . Then (h_1, \dots, h_k) is a partial coordinate system over $R_{\mathfrak{m}}$, which means that there are $f_{k+1}, \dots, f_n \in R_{\mathfrak{m}}[X]$ such that $F := (h_1, \dots, h_k, f_{k+1}, \dots, f_n)$ is a polynomial automorphism of $R_{\mathfrak{m}}^n$. The f_i can

be chosen in such a way that they have no constant part. Then $\det JF \in R_{\mathfrak{m}}[X]^*$ and hence substituting $X_1 := 0, \dots, X_n := 0$ gives

$$\begin{vmatrix} r_{11} & \dots & r_{1n} \\ \vdots & & \vdots \\ r_{k1} & \dots & r_{kn} \\ * & \dots & * \\ \vdots & & \vdots \\ * & \dots & * \end{vmatrix} = \det J(F[X := 0]) = (\det JF)[X := 0] \in R_{\mathfrak{m}}^*.$$

In particular, the matrix $(r_{ij})_{ij}$ represents a surjective $R_{\mathfrak{m}}$ -module homomorphism from $R_{\mathfrak{m}}^n$ to $R_{\mathfrak{m}}^k$.

Because this holds for every maximal ideal of R , it follows that the matrix $(r_{ij})_{ij}$ represents a surjective R -module homomorphism from R^n to R^k . Now R is Hermite, which implies that the matrix $(r_{ij})_{ij}$ can be extended to an invertible square matrix M over R (see [Lam78], Corollary 4.5). Viewing this matrix M as a polynomial automorphism of $R[X]$ and applying its inverse to the polynomials h_i , it follows that one can assume that (h_1, \dots, h_k) is of the form $(X_1 + \text{h.o.t.}, \dots, X_k + \text{h.o.t.})$. By Lemma 3.7.4, (h_1, \dots, h_k) then is a nice coordinate system in $R_{\mathfrak{m}}[X]$, for every $\mathfrak{m} \in \text{Max}(R)$. Now apply Theorem 3.7.10. \square

3.8 Cancellation Problem in two variables

As was already promised in Section 1.3, this section will prove that the Cancellation Problem has an affirmative answer in two variables over an arbitrary \mathbb{Q} -algebra. This result is needed in the next section.

Lemma 3.8.1. *Let R be a reduced \mathbb{Q} -algebra and let D be a locally nilpotent R -derivation on $R[X, Y]$. Then $\text{div}(D) = 0$. In other words, there is an $f \in R[X, Y]$ such that $D = f_Y \partial_X - f_X \partial_Y$.* \square

Proof. The case that R is a field is well-known. See, for instance, Proposition 1.3.51 of [Ess00]. In the general case of a reduced \mathbb{Q} -algebra, consider a prime ideal \mathfrak{p} of R . Then, by the field case, $\text{div}(D) = 0$ in $Q(R/\mathfrak{p})[X]$. Therefore $\text{div}(D) \in \bigcap_{\mathfrak{p} \in \text{Spec}(R)} \mathfrak{p} = \eta$. Because R is reduced $\eta = (0)$ and hence $\text{div}(D) = 0$. \square

Theorem 3.8.2. *Let R be a \mathbb{Q} -algebra and let D be a locally nilpotent R -derivation on $R[X, Y]$. Assume that D has a slice $s \in R[X, Y]$. Then $R[X, Y]^D \cong_R R^{[1]}$. In other words, s is a coordinate.*

Proof. Let η be the nilradical of R , write $\bar{R} := R/\eta$, and let \bar{D} be the \bar{R} -derivation on $\bar{R}[X, Y]$ induced by D . Because D is locally nilpotent, so is \bar{D} . By Lemma 3.8.1, there is an $f \in R[X, Y]$ such that $\bar{D} = f_Y \partial_X - f_X \partial_Y$. Because D has a slice, so has \bar{D} and therefore $1 \in (\bar{D}(X), \bar{D}(Y))$. Now, by Theorem 1.3.4 and Theorem 1.1.2, $\bar{R}[X, Y] = \bar{R}[X, Y]^D[\bar{s}] = \bar{R}[\bar{f}, \bar{s}]$, which means that (\bar{f}, \bar{s}) is an polynomial automorphism over \bar{R} . From Theorem 3.5.2 it follows that (f, s) is an polynomial automorphism over R . So s is a coordinate and $R[X, Y]^D = R[f]$. \square

3.9 s is a slice of some locally nilpotent derivation

Reduced Property 3.9.1. *Let $s \in R[X]$ and assume that there is some locally nilpotent derivation $E \in \text{Der}_{\bar{R}}(\bar{R}[X])$ such that $E(\bar{s}) = 1$. Then there exists a locally nilpotent derivation $D \in \text{Der}_R(R[X])$ such that $D(s) = 1$.*

Proof. Let $D_0 \in \text{Der}_R(R[X])$ be such that $\bar{D}_0 = E$. Then $D_0(s) = 1 - f$ for some $f \in \eta[X]$. Because $f \in \eta[X]$, there is an $m \in \mathbb{N}$ such that $f^m = 0$. Now let $D \in \text{Der}_R(R[X])$ be the derivation

$$D := \frac{1}{1-f} D_0 = (1 + f + f^2 + \cdots + f^{m-1}) D_0.$$

Then $D(s) = 1$. Furthermore, $\bar{D} = \bar{D}_0 = E$ and because E is locally nilpotent, D is locally nilpotent as well, by Property 3.2.1. \square

Prime Property 3.9.2. *Assume that R is a \mathbb{Q} -algebra and let $s \in R[X_1, X_2]$. Assume that for every $\mathfrak{p} \in \text{Spec}(R)$ there is some locally nilpotent R/\mathfrak{p} -derivation on $R/\mathfrak{p}[X_1, X_2]$ which has s as a slice. Then there exists a locally nilpotent R -derivation on $R[X_1, X_2]$ which has s as a slice.*

Proof. By Theorem 4.5.1 and Lemma 3.6.5, being a slice is equivalent to being a coordinate (in two variables over a \mathbb{Q} -algebra). Hence the claim follows from Property 3.6.7. \square

Local Property 3.9.3. *Assume that R is a \mathbb{Q} -algebra and let $s \in R[X_1, X_2]$. Assume that for every $\mathfrak{p} \in \text{Spec}(R)$ there is some locally nilpotent $R_{\mathfrak{p}}$ -derivation on $R_{\mathfrak{p}}[X_1, X_2]$ which has s as a slice. Then there exists a locally nilpotent R -derivation on $R[X_1, X_2]$ which has s as a slice.*

Proof. Just as the proof of Property 3.9.2, using Property 3.6.10 instead of Property 3.6.7. \square

Residual Property 3.9.4. Assume that R is a \mathbb{Q} -algebra and let $s \in R[X_1, X_2]$. Assume that for every $\mathfrak{p} \in \text{Spec}(R)$ there is some locally nilpotent $k_{\mathfrak{p}}$ -derivation on $k_{\mathfrak{p}}[X_1, X_2]$ which has s as a slice. Then there exists a locally nilpotent R -derivation on $R[X_1, X_2]$ which has s as a slice.

Proof. Just as the proof of Property 3.9.2, using Property 3.6.8 instead of Property 3.6.7. \square

Counterexample 3.9.5 (Quotient Property). The property “Being a slice of some locally nilpotent derivation” is not a quotient property: the example $s := TX_1 \in R[X_1]$ over the ring $R := \mathbb{C}[T]$ that has appeared before is also here a counterexample.

3.10 F is an embedding

Consider a polynomial map $F = (f_1, \dots, f_m): R^n \rightarrow R^m$, with each f_i an element of $R[X] := R[X_1, \dots, X_n]$ and $m \geq n$. Such a polynomial map is called an *embedding* if $R[f_1, \dots, f_m] = R[X]$. Embeddings are the subject of Chapter 5 of this thesis; this section and the next already study some reduction properties of embeddings.

Lemma 3.10.1. Let $F = (f_1, \dots, f_m): R^n \rightarrow R^m$ be a polynomial map. Let \mathfrak{a} and \mathfrak{b} be ideals of R and assume that $F/\mathfrak{a}: (R/\mathfrak{a})^n \rightarrow (R/\mathfrak{a})^m$ are embeddings. Then $F/\mathfrak{a}\mathfrak{b}: (R/\mathfrak{a}\mathfrak{b})^n \rightarrow (R/\mathfrak{a}\mathfrak{b})^m$ is an embedding.

Proof. For each $i \in \{1, \dots, n\}$, let $p_i \in R[Y] := R[Y_1, \dots, Y_m]$ be a polynomial such that $p'_i := p_i(f_1, \dots, f_m) - X_i \in \mathfrak{a}[X]$. Similarly, let $q_i \in R[Y]$ be a polynomial such that $q'_i := q_i(f_1, \dots, f_m) - X_i \in \mathfrak{b}[X]$. Then

$$\begin{aligned} X_i &= p_i(f_1, \dots, f_m) - p'_i(X_1, \dots, X_n) \\ &= p_i(F) - p'_i(q_1(f) - q'_1(X), \dots, q_n(f) - q'_n(X)) \\ &\equiv p_i(F) - p'_i(q_1(f), \dots, q_n(f)) \pmod{\mathfrak{a}\mathfrak{b}[X]}, \end{aligned}$$

since the coefficients of p'_i are elements of \mathfrak{a} and those of q'_1, \dots, q'_n are elements of \mathfrak{b} . \square

Reduced Property 3.10.2. Let $F = (f_1, \dots, f_m): R^n \rightarrow R^m$ be a polynomial map. Assume that $\bar{F}: \bar{R}^n \rightarrow \bar{R}^m$ is an embedding. Then F is an embedding.

Proof. Because $\bar{R}[\bar{f}_1, \dots, \bar{f}_m] = \bar{R}[X]$, there are polynomials $p_1, \dots, p_n \in R[Y] := R[Y_1, \dots, Y_m]$ such that $p'_i(X) := p_i(f_1, \dots, f_m) - X_i \in \eta[X]$ for all $i \in \{1, \dots, n\}$. Looking at the subring of R generated by the coefficients of f_1, \dots, f_m , and p_1, \dots, p_n , it is once again possible to assume that R is Noetherian.

By induction on e it will now follow from the previous lemma that F is an embedding modulo η^e for all $e \in \mathbb{N}^*$. Because R is Noetherian, its nilradical is finitely generated and hence there is an $e \in \mathbb{N}$ such that $\eta^e = (0)$. Using this e , one sees that F is an embedding. \square

Remark 3.10.3. Taking $m = n$ in the above property, this gives another way to proof that “being invertible” is a reduced property of a polynomial endomorphism (Property 3.5.2).

Prime Property 3.10.4. Assume that R is Noetherian. Let $F = (f_1, \dots, f_m) : R^n \rightarrow R^m$ be a polynomial map. Assume that $F/\mathfrak{p} : (R/\mathfrak{p})^n \rightarrow (R/\mathfrak{p})^m$ is an embedding for all $\mathfrak{p} \in \text{Spec}(R)$. Then F is an embedding.

Proof. Since R is Noetherian, its nilradical η is a finite intersection of prime ideals. Using the assumption and repeated application of Lemma 3.10.1, it follows that F is an embedding modulo η . By Property 3.10.2, F is an embedding. \square

Local Property 3.10.5. Let $F = (f_1, \dots, f_m) : R^n \rightarrow R^m$ be a polynomial map. Assume that $F_{\mathfrak{m}} : R_{\mathfrak{m}}^n \rightarrow R_{\mathfrak{m}}^m$ is an embedding for all $\mathfrak{m} \in \text{Max}(R)$. Then F is an embedding.

Proof. Since $R[F]_{\mathfrak{m}} = R_{\mathfrak{m}}[F] = R_{\mathfrak{m}}[X] = R[X]_{\mathfrak{m}}$ for all $\mathfrak{m} \in \text{Max}(R)$, it follows, for example from Proposition 3.9 of [AM69], that $R[F] = R[X]$. \square

Counterexample 3.10.6 (Residual Property). Let R be the ring $\mathbb{C}[T]$ and consider the polynomial map $F : R \rightarrow R^2$ given by $F := (X_1 + TX_1^2, T^2X_1)$. One immediately sees that $F_{\mathfrak{p}} : k_{\mathfrak{p}} \rightarrow k_{\mathfrak{p}}^2$ is an embedding for each $\mathfrak{p} \in \text{Spec}(R)$: at the prime ideals (0) and (T) , the first component equals X_1 ; at the prime ideals $(T - c)$, $c \in \mathbb{C}^*$, the second component equals cX_1 .

F itself, however, is not an embedding. One can see this by computing a Gröbner basis of the ideal $(X - (X_1 + TX_1^2), Y - T^2X_1)$ of $\mathbb{C}[X, Y, T, X_1]$ with respect to the lexicographical ordering $X_1 > T > Y > X$. The reduced Gröbner basis is $\{X_1^2Y + X_1YX - X_1 - TX^2 + X, X_1T + X_1Y - TX, X_1Y^2 + Y^2X - TYX - Y, T^3X - TY - Y^2\}$ and the normal form of X_1 with respect to this Gröbner basis is just X_1 itself. Since this is not an element of $\mathbb{C}[T, X, Y]$, it follows that X_1 is not an element of the algebra $\mathbb{C}[T, f_1, f_2]$. See, for instance, [BW93].

Counterexample 3.10.7 (Quotient Property). Consider the ring $R := \mathbb{C}[T]$ and $f_1 := TX_1 \in R[X_1]$. Then $K[f_1] = K[X_1]$, but $R[f_1] \neq R[X_1]$. This shows that “being an embedding” is not a quotient property.

3.11 F is rectifiable

Consider an embedding $F = (f_1, \dots, f_m): R^n \rightarrow R^m$, i.e., a polynomial map with $R[F] = R[X]$. Such an embedding is called *rectifiable* if there is a polynomial automorphism $G = (g_1, \dots, g_m)$ of R^m such that $G \circ F = (X_1, \dots, X_n, 0, \dots, 0)$. See also Chapter 5.

Reduced Property 3.11.1. Let $F = (f_1, \dots, f_m): R^n \rightarrow R^m$ be an embedding and assume that $\bar{F}: \bar{R}^n \rightarrow \bar{R}^m$ is rectifiable. Then F is rectifiable as well.

Proof. With the same kind of argument as before, one sees that R can be assumed to be Noetherian.

By induction on e it will now follow that for every $e \in \mathbb{N}^*$ there is a polynomial automorphism $G: R^m \rightarrow R^m$ such that $GF = (X_1, \dots, X_n, *, \dots, *) \pmod{\eta^e[X]}$.

$e = 1$: By assumption, there is a polynomial map $G: R^m \rightarrow R^m$ such that $\bar{G}: \bar{R}^m \rightarrow \bar{R}^m$ is invertible and $\bar{G}\bar{F} = (X_1, \dots, X_n, 0, \dots, 0)$. By Property 3.5.2, G itself is invertible.

$e > 1$: Using the induction hypothesis, let $G': R^m \rightarrow R^m$ be an invertible polynomial map such that

$$G' \circ F = (X_1, \dots, X_n, *, \dots, *) \pmod{\eta^e[X]},$$

say $p_1(X), \dots, p_n(X) \in \eta^e[X]$ are such that

$$G' \circ F = (X_1 + p_1(X), \dots, X_n + p_n(X), *, \dots, *).$$

Now let $H: R^m \rightarrow R^m$ be the polynomial map defined by

$$H := (X_1 - p_1(X), \dots, X_n - p_n(X), X_{n+1}, \dots, X_m).$$

Because $\bar{H} = (X_1, \dots, X_m)$, H is invertible by Property 3.5.2. Now take

$G := H \circ G'$. Then

$$\begin{aligned} G \circ F &= H \circ G' \circ F \\ &= (X_1 + p_1(X) - p_1(X_1 + p_1(X), \dots, p_n(X_1 + p_n(X))), \dots, \\ &\quad X_n + p_n(X) - p_n(X_1 + p_1(X), \dots, p_n(X_1 + p_n(X))), \\ &\quad *, \dots, *) \\ &\equiv (X_1, \dots, X_n, *, \dots, *) \pmod{\eta^{e+1}[X]}. \end{aligned}$$

This last equivalence even holds modulo $\eta^{2e}[X]$, since the coefficients of each p_i are elements of η^e .

Now because R is Noetherian, there is an $e \in \mathbb{N}^*$ such that $\eta^e = 0$. It follows that there is a polynomial automorphism $G: R^m \rightarrow R^m$ such that $GF = (X_1, \dots, X_n, *, \dots, *)$. This is enough to show that F is rectifiable. \square

Counterexample 3.11.2 (Local, Residual, and Quotient Property). Let the ring R be given by $R := \mathbb{R}[a, b, c]/(a^2 + b^2 + c^2 - 1)$ and let F be the embedding $F := (aX_1, bX_1, cX_1): R \rightarrow R^3$.

If \mathfrak{p} is a prime ideal of R , then the residue classes \bar{a} , \bar{b} , and \bar{c} cannot be all elements of \mathfrak{p} . Say $\bar{a} \notin \mathfrak{p}$. Then \bar{a} is invertible in $R_{\mathfrak{p}}$ (and in $k_{\mathfrak{p}}$) and hence F is rectifiable over $R_{\mathfrak{p}}$ (and over $k_{\mathfrak{p}}$). Also, R is a domain and over the quotient field of R , F is obviously rectifiable.

Over R itself, F is not rectifiable. For suppose that $G: R^3 \rightarrow R^3$ is a polynomial automorphism such that $G(aX_1, bX_1, cX_1) = (X_1, 0, 0)$. Taking derivatives at $X_1 = 0$, it follows that

$$(JG)(0) \begin{pmatrix} \bar{a} \\ \bar{b} \\ \bar{c} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

So the inverse of $(JG)(0)$ would be an invertible square matrix extending the unimodular row $(\bar{a}, \bar{b}, \bar{c})$. Such a matrix, however, does not exist. See also Example 2.2.8.

3.12 Summary

Table on page 71 summarizes the results of this chapter. A \checkmark means that the property holds; the following number is the reference to the property and between brackets are additional conditions under which the property holds. A $-$ means that the property does not hold.

	reduced	prime	local	residual	quotient
$\mathfrak{a} = R[X]$	✓:3.1.2	✓:3.1.3	✓:3.1.4	✓:3.1.6	–:3.1.5
D loc.nilp.	✓:3.2.1	✓:3.2.3(R Noeth) ✓:3.2.7(R \mathbb{Q} -alg, $n = 2$)	✓:3.2.2	✓:3.2.5(R Noeth) ✓:3.2.8(R \mathbb{Q} -alg, $n = 2$)	✓:3.2.4
D has slice	✓:3.3.4(R \mathbb{Q} -alg, D loc.nilp.)	✓:3.3.5(R Noeth \mathbb{Q} -alg, D loc.nilp.)	✓:3.3.7	–:3.3.8	–:3.3.9
$R[X]^\nu$ fin.gen.	–:3.4.1(R \mathbb{Q} -alg)	–:3.4.1(R \mathbb{Q} -alg)		–:3.4.2(R \mathbb{Q} -alg)	–:3.4.2(R \mathbb{Q} -alg)
F invertible	✓:3.5.2	✓:3.5.4	✓:3.5.3	✓:3.5.7	✓:3.5.5($\det(JF) = 1$) –:3.5.6
f coordinate	✓:3.6.1	✓:3.6.7(R \mathbb{Q} -alg, $n = 2$)	✓:3.7.11(R Herm \mathbb{Q} -dom) ✓:3.6.10(R \mathbb{Q} -alg, $n = 2$) –:3.6.3	✓:3.6.8(R \mathbb{Q} -alg, $n = 2$) –:3.6.3	–:3.6.4
s is slice	✓:3.9.1	✓:3.9.2(R \mathbb{Q} -alg, $n = 2$)	✓:3.9.3(R \mathbb{Q} -alg, $n = 2$)	✓:3.9.4(R \mathbb{Q} -alg, $n = 2$)	–:3.9.5
F is embedding	✓:3.10.2	✓:3.10.4(R Noeth)	✓:3.10.5	–:3.10.6	–:3.10.7
F is rectifiable	✓:3.11.1		–:3.11.2	–:3.11.2	–:3.11.2

Table 3.1: Summary of results

Chapter 4

Applications

The results of the previous chapter can be used to generalise known results over fields to other rings.

The first three sections of this chapter use this technique to study coordinates in two variables over arbitrary \mathbb{Q} -algebras. They give several criteria to recognise if a polynomial in two variables is a coordinate. In particular, the Abhyabkar-Moh-Suzuki Theorem ([AM75], [Suz74]) is generalised to arbitrary \mathbb{Q} -algebras.

The fourth section makes some remarks about how these criteria do or do not generalise to more than two variables.

The last section of this chapter generalises Miyanishi's result on the kernel of locally nilpotent derivations in three variables over a field of characteristic zero (Theorem 1.3.5) to polynomial rings in three variables over a Dedekind domain containing \mathbb{Q} . This generalisation is then used to show that the Cancellation Problem has an affirmative answer in four variables over a field of characteristic zero for a large class of locally nilpotent derivations, including the triangular ones.

4.1 Coordinates

Let R be a ring and let $f \in R[X] := R[X_1, \dots, X_n]$ be a coordinate. Say $f_2, \dots, f_n \in R[X]$ are such that $F := (f, f_2, \dots, f_n)$ is an invertible polynomial map. Then the determinant of the Jacobian matrix $J(f, f_2, \dots, f_n)$ is a unit in $R[X]$. Taking the Laplace development of this determinant according to its first row, it follows that 1 belongs to the ideal generated by the partial derivatives of f . This motivates the following definition.

Definition 4.1.1. A polynomial $f \in R[X]$ is called *unimodular* if the ideal of $R[X]$ generated by its partial derivatives f_{X_1}, \dots, f_{X_n} is the unit ideal.

So, the argument given above shows that every coordinate of $R[X]$ is unimodular. Conversely, if f is unimodular in $R[X]$, then already in the case that $n = 2$ the polynomial f does not have to be a coordinate in $R[X]$. For example, take the unimodular polynomial $f := X_1 + X_1^2 X_2$ in $R[X_1, X_2]$. This is not a coordinate, because it is reducible. Proposition 3.6.6, in fact, already described exactly which additional condition is necessary and sufficient to guarantee that a polynomial in two variables over a \mathbb{Q} -algebra R is a coordinate: a polynomial $f \in R[X_1, X_2]$ is a coordinate if and only if it is unimodular and the derivation $f_{X_2} \partial_{X_1} - f_{X_1} \partial_{X_2}$ is locally nilpotent.

If R is not a \mathbb{Q} -algebra, then this does not have to be true: take $R := \mathbb{Z}[T]/(2T)$ and $f := X_1 - TX_1^2$. Then $(f_{X_1}, f_{X_2}) = (1)$ and the derivation $f_{X_2} \partial_{X_1} - f_{X_1} \partial_{X_2} = -\partial_{X_2}$ is locally nilpotent. Furthermore, f is not a coordinate in $R[X_1]$. This is just another way of saying that the polynomial map (f) in one variable over R is not invertible: its formal inverse is $(X_1 + \sum_{i=1}^{\infty} \bar{T}^{2^i} X_1^{2^i})$. See [Ess00], Warning 1.1.17. From the next lemma it then follows that f is not a coordinate in $R[X_1, X_2]$ either.

Lemma 4.1.2. *Let $f_1(X), \dots, f_n(X) \in R[X] := R[X_1, \dots, X_n]$ and write $Y := Y_1, \dots, Y_m$. If there are $g_1(X, Y), \dots, g_m(X, Y) \in R[X, Y]$ such that $R[f_1, \dots, f_n, g_1, \dots, g_m] = R[X, Y]$, then $R[f_1, \dots, f_n] = R[X]$.*

Proof Let $g_1(X, Y), \dots, g_m(X, Y) \in R[X, Y]$ be polynomials such that

$$R[f_1, \dots, f_n, g_1, \dots, g_m] = R[X, Y].$$

This means that (f, g) is an R -automorphism of $R[X, Y]$. Let $(h_1(X, Y), \dots, h_{m+n}(X, Y))$ be its inverse. Then, in particular,

$$X_i = f_i(h_1(X, Y), \dots, h_n(X, Y))$$

for all $i \in \{1, \dots, n\}$. Substituting $Y_j := 0$ for all $j \in \{1, \dots, m\}$ shows that $R[f_1, \dots, f_n] = R[X]$. \square

The Abhyankar-Moh Theorem is commonly formulated in the following way ([AM75]).

Theorem 4.1.3 (Abhyankar-Moh Theorem). *Let k be a field of characteristic zero and let $f, g \in k[T]$. Assume that $k[f, g] = k[T]$. Then $\deg f$ divides $\deg g$ or conversely.* \square

Another way to formulate this result is the following. In this form, it was also discovered by Suzuki in [Suz74].

Theorem 4.1.4 (Abhyankar-Moh-Suzuki Theorem). *Let k be a field of characteristic zero and let $f \in k[X, Y]$. Assume that $k[X, Y]/(f) \cong_k k^{[1]}$. Then f is a coordinate in $k[X, Y]$.* \square

As a second criterion for recognising coordinates in two variables, this theorem can now be generalised as follows.

Theorem 4.1.5 (Generalised Abhyankar-Moh-Suzuki Theorem). *Let R be a \mathbb{Q} -algebra and let $f \in R[X, Y]$. Assume that $R[X, Y]/(f) \cong R^{[1]}$. Then f is a coordinate in $R[X, Y]$.*

Proof Let D be the derivation $f_Y \partial_X - f_X \partial_Y$ on $R[X, Y]$ and consider a prime ideal \mathfrak{p} of R . Then $k_{\mathfrak{p}}[X, Y]/(f_{\mathfrak{p}}) \cong k_{\mathfrak{p}}^{[1]}$. Since R is a \mathbb{Q} -algebra, $k_{\mathfrak{p}}$ is a field of characteristic 0. So the Abhyankar-Moh-Suzuki Theorem implies that $f_{\mathfrak{p}}$ is a coordinate over $k_{\mathfrak{p}}$ in $k_{\mathfrak{p}}[X, Y]$. Now Theorem 3.6.8 implies that f itself is a coordinate over R . \square

4.2 Coordinates in two variables under ring extensions

The main result of this section, Theorem 4.2.2, gives a criterion which decides if a polynomial in $R[X_1, X_2]$ which is a coordinate in a larger polynomial ring $S[X_1, \dots, X_n]$ for some $n \geq 2$ and some ring extension $R \subseteq S$, is a coordinate in $R[X_1, X_2]$.

Lemma 4.2.1. *Let $K \subseteq L$ be a field extension and let \mathfrak{a} be an ideal of the polynomial ring $K[X_1, \dots, X_n]$. Let \mathfrak{b} be the ideal of $L[X_1, \dots, X_n]$ generated by the elements of \mathfrak{a} . If $1 \in \mathfrak{b}$, then $1 \in \mathfrak{a}$.*

Proof Left to the reader. \square

Theorem 4.2.2. *Let $R \subseteq S$ be an extension of \mathbb{Q} -algebras, $f \in R[X_1, X_2]$, and let $n \geq 2$. Assume that (at least) one of the following two conditions holds:*

- (a) *f is unimodular;*
- (b) *$R \subseteq S$ satisfies the going-up property, i.e., for every $\mathfrak{p} \in \text{Spec}(R)$ there is a $\mathfrak{q} \in \text{Spec}(S)$ such that $\mathfrak{p} = \mathfrak{q} \cap R$.*

Then the following four statements are equivalent:

1. *f is a coordinate in $R[X_1, X_2]$;*
2. *f is a coordinate in $R[X_1, \dots, X_n]$;*

3. f is a coordinate in $S[X_1, \dots, X_n]$;
4. $S[X_1, \dots, X_n]/(f) \cong_S S^{[n-1]}$.

Proof. It is enough to show $4 \Rightarrow 1$. Let $\mathfrak{q} \in \text{Spec}(S)$. Then

$$k_{\mathfrak{q}}[X_1, \dots, X_n]/(f_{\mathfrak{q}}) \cong_{k_{\mathfrak{q}}} k_{\mathfrak{q}}^{[n-1]},$$

so $k_{\mathfrak{q}}[X_1, X_2]/(f_{\mathfrak{q}})[X_3, \dots, X_n] \cong_{k_{\mathfrak{q}}} k_{\mathfrak{q}}^{[n-1]}$. It now follows from Corollary 2.8 of [AHE72] that $k_{\mathfrak{q}}[X_1, X_2]/(f_{\mathfrak{q}}) \cong_{k_{\mathfrak{q}}} k_{\mathfrak{q}}^{[1]}$ and hence the Abhyankar-Moh-Suzuki Theorem implies that $f_{\mathfrak{q}}$ is a coordinate in $k_{\mathfrak{q}}[X_1, X_2]$.

So, by Property 3.6.8, f is a coordinate in $S[X_1, X_2]$. Consequently, the derivation $D := f_{X_2}\partial_{X_1} - f_{X_1}\partial_{X_2}$ is locally nilpotent on $S[X_1, X_2]$ and hence on $R[X_1, X_2]$.

- (a) Firstly, assume that condition (a) is satisfied. Because f is unimodular, Proposition 3.6.6 now implies that f is a coordinate in $R[X_1, X_2]$.
- (b) Secondly, assume that condition (b) is satisfied. It is enough to show that $(f_{\mathfrak{p}X_1}, f_{\mathfrak{p}X_2}) = k_{\mathfrak{p}}[X_1, X_2]$ for all prime ideals \mathfrak{p} of R , because then Property 3.1.6 implies that f is unimodular in $R[X_1, X_2]$ and Proposition 3.6.6 can once again be applied.

So let \mathfrak{p} be a prime ideal of R and choose a prime ideal \mathfrak{q} of S such that $\mathfrak{q} \cap R = \mathfrak{p}$. So $R/\mathfrak{p} \subseteq S/\mathfrak{q}$ and therefore $k_{\mathfrak{p}} \subseteq k_{\mathfrak{q}}$. Furthermore $f_{\mathfrak{q}}$ is a coordinate in $k_{\mathfrak{q}}[X_1, X_2]$ and therefore $(f_{\mathfrak{q}X_1}, f_{\mathfrak{q}X_2}) = k_{\mathfrak{q}}[X_1, X_2]$. Applying Lemma 4.2.1 to the field extension $k_{\mathfrak{p}} \subseteq k_{\mathfrak{q}}$ gives $(f_{\mathfrak{p}X_1}, f_{\mathfrak{p}X_2}) = k_{\mathfrak{p}}[X_1, X_2]$. \square

Taking $S = R$ in this theorem gives the following generalisation of Theorem 4.1.5.

Corollary 4.2.3. *Assume that R is a \mathbb{Q} -algebra. Let $f \in R[X_1, X_2]$ and $n \geq 2$. Then f is a coordinate in $R[X_1, X_2]$ if and only if f is a coordinate in $R[X_1, \dots, X_n]$ if and only if $R[X_1, \dots, X_n]/(f) \cong_R R^{[n-1]}$.* \square

Also note that if neither of the conditions (a) and (b) is satisfied, then the implication $4 \Rightarrow 1$ of Theorem 4.2.2 is obviously false: take $R := \mathbb{C}[T]$, $S := \mathbb{C}[T, T^{-1}]$, and $f := TX_1$.

One might think that the condition $S^* \cap R = R^*$ is the only obstruction to the implication $4 \Rightarrow 1$. However, the example $R := \mathbb{Q}[a, b]$, $S := \mathbb{Q}[a, b, c, d]/(ab - bc - 1)$, $f := aX_1 + bX_2$ shows that this is not the case. Here $S^* \cap R = R^*$, as one easily verifies, and f is a coordinate in $S[X_1, X_2]$, but not in $R[X_1, X_2]$ (since f is not unimodular in $R[X_1, X_2]$).

4.3 Endomorphisms sending linear coordinates to coordinates

Assume that R is a \mathbb{Q} -algebra. The main result of [CE00] asserts that if R is a field, then every R -endomorphism of $R[X, Y]$ sending all linear coordinates to coordinates is an R -automorphism of $R[X, Y]$. This section generalises this result to arbitrary \mathbb{Q} -algebras.

Definition 4.3.1. A linear polynomial $aX + bY \in R[X, Y]$ is called an *elementary linear coordinate* if $a = 1$ or $b = 1$.

Theorem 4.3.2. Let $F: R^2 \rightarrow R^2$ be a polynomial map and let $\varphi := F^* \in \text{End}_R(R[X, Y])$ be the corresponding map of algebras. Assume that φ sends every elementary linear coordinate to a coordinate in $R[X, Y]$. Then F is invertible.

Proof. Since $\varphi(X)$ is a coordinate of $R[X, Y]$, there exists an R -automorphism ψ of $R[X, Y]$ with $\psi(X) = \varphi(X)$. So $\psi^{-1}\varphi$ is an R -endomorphism of $R[X, Y]$ which sends every elementary linear coordinate of $R[X, Y]$ to a coordinate of $R[X, Y]$ and which sends X to X . It is obviously enough to show that $\psi^{-1}\varphi$ is an R -automorphism and, hence, it is possible to assume, replacing φ by $\psi^{-1}\varphi$, that φ is of the form (X, g) for some $g \in R[X, Y]$.

By Property 3.5.7, it suffices to show that $(X, g_{\mathfrak{p}})$ is a $k_{\mathfrak{p}}$ -automorphism of $k_{\mathfrak{p}}[X, Y]$, for all $\mathfrak{p} \in \text{Spec}(R)$.

So let \mathfrak{p} be a fixed prime ideal of R . Since $g - cX = \varphi(Y - cX)$ is a coordinate for all $c \in R$, it follows that $g - \bar{c}X$ is a coordinate in $k_{\mathfrak{p}}[X, Y]$ for all $c \in R$. Here $\bar{c} := c + \mathfrak{p} \in R/\mathfrak{p} \subseteq k_{\mathfrak{p}}$. Hence, taking for c the coefficient of the monomial X appearing in g , it follows that $\deg_Y g > 0$. Write

$$g = g_n Y^n + g_{n-1} Y^{n-1} + \cdots + g_0,$$

with each $g_i \in k[X]$, $n \geq 1$ and $g_n \neq 0$. Since $g = \varphi(Y)$ is a coordinate in $R[X, Y]$, g is also a coordinate when considered as an element of $k_{\mathfrak{p}}[X, Y]$. Then by [AE90], Corollary 1.4 or [Ess00], Corollary 3.3.7, it follows that $g_n \in k_{\mathfrak{p}}^*$. Consequently, it is enough to show that $n = 1$, for then $\varphi_{\mathfrak{p}} = (X, g_1 Y + g_0)$, which is obviously a $k_{\mathfrak{p}}$ -automorphism of $k_{\mathfrak{p}}[X, Y]$.

Assume that $n \geq 2$. Replacing g by $g_n^{-1}g$ it is possible to assume that $g_n = 1$. Furthermore, replacing Y by $Y - n^{-1}g_{n-1}$ it is possible to assume that $g_{n-1} = 0$.

From now on, the argument follows the proof of Theorem 1.1 of [CE00]. Put $D := g_Y \partial_X - g_X \partial_Y$. Since $\mathbb{Q} \subseteq k_{\mathfrak{p}}$, it follows that $(D + q \partial_Y)^n(X) \in k_{\mathfrak{p}}$ for all $q \in \mathbb{Q}$. Then by Lemma 1.3 of [CE00] (using the fact that \mathbb{Q} is infinite) it follows that the polynomial $h(t) := (D + t \partial_Y)^n(X)$ is an element of $k_{\mathfrak{p}}[t]$. So, in particular,

the coefficient of t^{n-2} of $h(t)$ belongs to k . Then by Proposition 2.1 of [CE00] one sees that $g_X \in k$. So $g = \lambda X + a(Y)$ for some $\lambda \in k$ and some $a(Y) \in k[Y]$. Since $g \in R[X, Y]$, the coefficients of g belong in fact to $\bar{R} := R/\mathfrak{p} \subseteq k$, i.e., $\lambda = \bar{c}$ for some $c \in R$. Then again using the fact that $g - \bar{c}X$ is a coordinate in $k[X, Y]$, we get that $a(Y)$ is a coordinate in $k[X, Y]$ and hence in $k[Y]$ (by Lemma 4.1.2). But this is a contradiction, since $\deg a(Y) = n \geq 2$. \square

4.4 Remarks on coordinates in dimensions greater than two

In the previous sections various results for coordinates in two variables were given. This section considers coordinates in more than two variables.

The Abhyankar-Sathaye Conjecture states the following: for $n \geq 3$, if k is a field of characteristic zero and $f \in k[X_1, \dots, X_n]$ is a polynomial such that $k[X]/(f) \cong_k k^{[n-1]}$, then f is a coordinate. The following special case of the conjecture was proven by Sathaye ([Sat76]) and Russell ([Rus76]).

Theorem 4.4.1. *Let k be a field and let $f(X, Y, Z)$ be a polynomial over k of the form*

$$f(X, Y, Z) = g(X, Y)Z + h(X, Y).$$

Assume that $k[X, Y, Z]/(f) \cong_k k^{[2]}$. Then f is a coordinate over $k[X]$. \square

In fact, the proof of this theorem even shows that, under the assumption that $k[X, Y, Z]/(f) \cong_k k^{[2]}$, the polynomial $g(X, Y)$ is in fact a polynomial in a coordinate, i.e. $g(X, Y) = \tilde{g}(c(X, Y))$ for some $\tilde{g}(T) \in k[T]$ and some coordinate $c(X, Y) \in k[X, Y]$. This suggests the following generalisation of this theorem.

Theorem 4.4.2. *Let R be an arbitrary \mathbb{Q} -algebra and let $f(X, Y, Z) \in R[X, Y, Z]$ be a polynomial of the form*

$$f(X, Y, Z) = g(c(X, Y))Z + h(X, Y),$$

for some coordinate $c(X, Y) \in R[X, Y]$ and some polynomial $g(T) \in R[T]$ and $h(X, Y) \in R[X, Y]$. Assume that $R[X, Y, Z]/(f) \cong_R R^{[2]}$. Then f is a coordinate over $R[X]$.

Proof. It is possible to assume, without loss of generality, that $c(X, Y) = X$. So $f = g(X)Z + h(X, Y)$.

Now let $\mathfrak{q} \subseteq R[X]$ be a prime ideal and take $\mathfrak{p} := \mathfrak{q} \cap R$. Since $R[X, Y, Z]/(f)$ is isomorphic, as an R -algebra, to $R^{[2]}$, also $k_{\mathfrak{p}}[X, Y, Z]/(f_{\mathfrak{p}}) \cong_{k_{\mathfrak{p}}} k_{\mathfrak{p}}^{[2]}$. By Theorem 4.4.1, $f_{\mathfrak{p}}$ is a coordinate over $k_{\mathfrak{p}}[X]$ in $k_{\mathfrak{p}}[X, Y, Z]$. Let $g \in k[X, Y, Z]$ be a polynomial such that $(f_{\mathfrak{p}}, g_{\mathfrak{p}}) \in \text{Aut}_{k_{\mathfrak{p}}[X]}(k_{\mathfrak{p}}[X, Y, Z])$ and let φ be the natural map from $k_{\mathfrak{p}}[X, Y, Z]$ to $(R[X]_{\mathfrak{q}}/\mathfrak{q}R[X]_{\mathfrak{q}})[Y, Z]$. Then

$$(f_{\mathfrak{q}}, g_{\mathfrak{q}}) = (\varphi(f_{\mathfrak{p}}), \varphi(g_{\mathfrak{p}})) \in \text{Aut}_{R[X]_{\mathfrak{q}}/\mathfrak{q}R[X]_{\mathfrak{q}}}(R[X]_{\mathfrak{q}}/\mathfrak{q}R[X]_{\mathfrak{q}})[Y, Z],$$

so $f_{\mathfrak{q}}$ is a coordinate over the residue field $R[X]_{\mathfrak{q}}/\mathfrak{q}R[X]_{\mathfrak{q}}$.

Hence, applying Property 3.6.8 to the ring $R[X]$ and the polynomial f , it follows that f is a coordinate over $R[X]$. \square

The following example shows that it is in general not true that a polynomial that is residually a polynomial in a coordinate, is itself a polynomial in a coordinate.

Example 4.4.3. Take $R := \mathbb{C}[T]$ and $f := X^2 + TY \in R[X, Y]$. Then $f_{\mathfrak{p}}$ is a coordinate in $k_{\mathfrak{p}}[X, Y]$ for every prime ideal $\mathfrak{p} \neq 0$ and $f_{(0)} = X^2$. The polynomial f itself is not a polynomial in a coordinate, though, since it is not unimodular.

Finally, this section exhibits several examples showing that most results from the previous sections do not hold for more than two variables.

Concerning a possible generalisation of Theorem 4.1.5: here almost nothing is known. Even in the case $n = 3$ and $R = \mathbb{C}$ it is still an open problem if $\mathbb{C}[X_1, X_2, X_3]/(f) \cong_{\mathbb{C}} \mathbb{C}^{[2]}$ implies that f is a coordinate in $\mathbb{C}[X_1, X_2, X_3]$. This is the 3-dimensional version of the Abhyankar-Sathaye Conjecture. Some special results are obtained in [Sat76], [Rus76], and [BD94]. See also Chapter 5, Section 3 of [Ess00].

Lemma 4.4.4. *Let R be a \mathbb{Q} -algebra and let D be a locally nilpotent R -derivation on $R[X_1, \dots, X_n]$ with a slice $s \in R[X_1, \dots, X_n]$. Then s is a coordinate when considered as an element of $R[X_1, \dots, X_{n+1}]$.*

Proof. The derivation D can be extended to a derivation on $R[X_1, \dots, X_{n+1}]$ simply by sending X_{n+1} to 0. This extension will be denoted by D as well. In order to show that s is a coordinate in $R[X_1, \dots, X_{n+1}]$, it is enough to show that $R[X_1, \dots, X_{n+1}]^D \cong_R R^{[n]}$, by Proposition 1.5.7. Now

$$\begin{aligned} R[X_1, \dots, X_{n+1}]^D &= R[X_1, \dots, X_n]^D[X_{n+1}] \\ &\cong_R R[X_1, \dots, X_n]^D[s] \\ &= R[X_1, \dots, X_n], \end{aligned}$$

using Theorem 1.1.2 twice. \square

This lemma can be used to show that Theorem 4.2.2 cannot be extended to higher dimensions.

Example 4.4.5. Take $R := \mathbb{R}[a, b, c]/(a^2 + b^2 + c^2 - 1)$, $D := a\partial_{X_1} + b\partial_{X_2} + c\partial_{X_3}$, and $s := aX_1 + bX_2 + cX_3$. Then s is not a coordinate (see Example 2.2.8), but it is a coordinate in $R[X_1, X_2, X_3, X_4]$.

Concerning Theorem 4.3.2, it was already shown in [MYZ97] that a similar result does not hold in dimensions greater than two.

Example 4.4.6. Let $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the polynomial map defined by $F := (X_1 + X_2X_3, X_2 - X_1X_3, X_3)$. Then $\det(J\varphi) = 1 + X_3^2$, so φ is not an \mathbb{R} -automorphism of $\mathbb{R}[X_1, X_2, X_3]$. On the other hand, it was shown in [MYZ97] that φ does send linear coordinates in $\mathbb{R}[X_1, X_2, X_3]$ to coordinates in $\mathbb{R}[X_1, X_2, X_3]$.

4.5 Dimension four

In [Sat83], Sathaye has proven the following characterisation of a polynomial ring in two variables over a discrete valuation ring containing \mathbb{Q} .

Theorem 4.5.1. *Let R be a discrete valuation ring containing \mathbb{Q} . Denote the unique maximal ideal of R by \mathfrak{m} , write K for the quotient field $Q(R)$ of R , and write k for the residue field R/\mathfrak{m} of R . Let A be a finitely generated R -domain and assume that $K \otimes_R A \cong_K K^{[2]}$ and that $k \otimes_R A \cong_k k^{[2]}$. Then $A \cong_R R^{[2]}$. \square*

In order to use this result, a lemma is needed on the behaviour of the kernel of a locally nilpotent derivation with a slice under tensoring.

Lemma 4.5.2. *Let $s \in R[X] := R[X_1, \dots, X_n]$ and let A be an R -algebra via the map $\varphi: R \rightarrow A$. Denote the induced map $R[X] \rightarrow A[X]$ by $\varphi_\#$. Then*

$$A \otimes_R R[X]/(sR[X]) \cong_A A[X]/(\varphi_\#(s)A[X])$$

In particular, if D is a locally nilpotent R -derivation on $R[X]$ and s is a slice of D , then

$$A \otimes_R R[X]^D \cong_A A[X]^{\tilde{D}},$$

where \tilde{D} denotes the extension of D to $A[X]$.

Proof. The following diagram is a commutative diagram of R -modules and R -module homomorphism in which the horizontal sequences are exact.

$$\begin{array}{ccccccc}
 sR[X] & \longrightarrow & R[X] & \longrightarrow & R[X]/sR[X] & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 A \otimes_R sR[X] & \longrightarrow & A \otimes_R R[X] & \longrightarrow & A \otimes_R R[X]/sR[X] & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \varphi_{\#}(s)A[X] & \longrightarrow & A[X] & \longrightarrow & A[X]/(\varphi_{\#}(s)A[X]) & \longrightarrow & 0
 \end{array}$$

The map $A \otimes_R sR[X] \rightarrow \varphi_{\#}(s)A[X]$ is surjective: take an element $\varphi_{\#}(s)f \in A[X]$ with $f \in A[X]$. Write $f = \sum_{\alpha} c_{\alpha} X_1^{\alpha_1} \dots X_n^{\alpha_n}$ with each $c_{\alpha} \in A$. Then $\varphi_{\#}(s)f$ is the image of $\sum_{\alpha} c_{\alpha} \otimes sX_1^{\alpha_1} \dots X_n^{\alpha_n}$. Also, the map $A \otimes_R R[X] \rightarrow A[X]$ is an isomorphism. Hence, by the Five Lemma, the map $A \otimes_R R[X]/sR[X] \rightarrow A[X]/(\varphi_{\#}(s)A[X])$ is an isomorphism. A priori this is an isomorphism of R -modules. However, since it is an A -module homomorphism, it is even an isomorphism of A -modules.

The second claim follows from the first one using Theorem 1.1.2. \square

Note that this lemma is false if D does not have slice. For instance, let K be some field, $R := K[Y]$, and consider $A := K$ as an R -module by sending elements of K to themselves and Y to 0. Let D be the locally nilpotent derivation $Y\partial_X$ on $R[X]$. Then $R[X]^D = R$, so $A \otimes_R R[X]^D = A = K$. However, the extension \tilde{D} of D to $A[X]$ is 0 and hence $A[X]^{\tilde{D}} = A[X]$.

Lemma 4.5.3. *Let R be a discrete valuation ring containing \mathbb{Q} and let D be a locally nilpotent R -derivation on $R[X, Y, Z]$ with a slice $s \in R[X, Y, Z]$. Then $R[X, Y, Z]^D \cong_R R^{[2]}$.*

Proof. Let k be the residue field of R and let K be the quotient field of R . Denote the extension of D to $K \otimes_R R[X, Y, Z] \cong_K K[X, Y, Z]$ by \tilde{D} . By Lemma 4.5.2 and Theorem 1.3.5 it follows that

$$K \otimes_R R[X, Y, Z]^D \cong_K K[X, Y, Z]^{\tilde{D}} \cong_K K^{[2]}.$$

In exactly the same way it follows that

$$k \otimes_R R[X, Y, Z]^D \cong_k k^{[2]}.$$

Hence, by Theorem 4.5.1, $R[X, Y, Z]^D \cong_R R^{[2]}$. \square

Now it is possible to show that the Cancellation Problem has an affirmative answer in three variables over a Dedekind domain containing \mathbb{Q} and, as a consequence, that it has an affirmative answer in four variables for a class of derivations including the triangular ones.

Theorem 4.5.4. *Let R be a Dedekind domain containing \mathbb{Q} and let D be a locally nilpotent R -derivation on $R[X, Y, Z]$ with a slice. Then $R[X, Y, Z]^D \cong_R R^{[2]}$.*

Proof. Let $s \in R[X, Y, Z]$ be a slice of D . Note that a unimodular row of length 2 is always extendible to an invertible square matrix and by Bass' Cancellation Theorem for Stably Free Modules ([Bas68], Theorem V.3.2; see also [Wei00], Theorem 1.3) every unimodular row of length at least 3 over a Noetherian ring of dimension one is extendible. In particular, R is Hermite. By Property 3.7.11 it is enough to show that s is a coordinate in $R_{\mathfrak{m}}[X, Y, Z]$ for every maximal ideal \mathfrak{m} of R .

So let \mathfrak{m} be a maximal ideal of R . Then $R_{\mathfrak{m}}$ is a discrete valuation ring. Because R contains \mathbb{Q} , $R_{\mathfrak{m}}$ contains \mathbb{Q} as well. Now Lemma 4.5.3 implies that $R_{\mathfrak{m}}[X, Y, Z]^D \cong_{R_{\mathfrak{m}}} R_{\mathfrak{m}}^{[2]}$. In other words, s is a coordinate in $R_{\mathfrak{m}}[X, Y, Z]$. \square

Corollary 4.5.5. *Let k be a field of characteristic zero and let D be a locally nilpotent k -derivation on $k[X, Y, Z, W]$ of the form*

$$D := a(X, Y, Z, W)\partial_X + b(X, Y, Z, W)\partial_Y + c(X, Y, Z, W)\partial_Z + d(W)\partial_W.$$

Assume that D has a slice. Then $k[X, Y, Z, W]^D \cong_k k^{[3]}$.

Proof. If $d(W) \neq 0$, then $d(W) \in k^*$, since D is locally nilpotent. So $d(W)^{-1}W$ is a slice of D . This slice is also a coordinate and hence $k[X]^D \cong_k k^{[3]}$. Otherwise, if $d(W) = 0$, apply Theorem 4.5.4 with $R = k[W]$. \square

Chapter 5

Embeddings

This chapter studies embeddings of a \mathbb{Q} -algebra R into R^n . It associates a locally nilpotent derivation to such an embedding and uses the kernel of this derivation to gather information about the embedding itself. The research in this chapter also appeared in [ER00b] and in [ER00a].

The first section introduces embeddings of a ring R into R^n . It also introduces the Embedding Problem, which asks whether every embedding of \mathbb{C} into \mathbb{C}^n is rectifiable. In the second section, embeddings from R into R^2 are studied and a characterisation of rectifiable embeddings is given. This section also shows that over an arbitrary \mathbb{Q} -algebra, not every embedding of R into R^2 is rectifiable, contrasting the case where $R = \mathbb{C}$. The third section associates a locally nilpotent derivation to a map from R to R^n and shows that such a map is an embedding if and only if the corresponding derivation has a slice. Section four looks at the case where R is a field of characteristic zero and shows that in that case, for $n \neq 3$, the Cancellation Problem has an affirmative answer for the derivations belonging to an embedding. The last section of this chapter constructs a candidate counterexample to the Cancellation Problem, the Linearisation Problem, and the Embedding Problem.

5.1 Embeddings and rectifiability

Let R be a ring, $r \in \{1, \dots, n\}$, and consider a polynomial map $\alpha: R^r \rightarrow R^n$ given by $\alpha := (f_1(T_1, \dots, T_r), \dots, f_n(T_1, \dots, T_r))$ for certain polynomials $f_1, \dots, f_n \in R[T] := R[T_1, \dots, T_r]$. Then α is called an *embedding of R^r in R^n* if $R[f_1, \dots, f_n] = R[T]$. Equivalently, α is called an embedding if the induced ring homomorphism $\alpha^*: R[X_1, \dots, X_n] \rightarrow R[T]$, which sends each X_i to f_i , is

surjective.

Proposition 5.1.1. *Let $\alpha: k^r \rightarrow k^n$ be an embedding over an infinite field k . The map α is an embedding if and only if $\text{Im}(\alpha)$ is a closed subset of k^n (in the Zariski topology) and $\alpha: k^r \rightarrow \text{Im}(\alpha)$ is an isomorphism of algebraic varieties over k .*

Proof Write $\alpha = (f_1(T), \dots, f_n(T))$ with each $f_i \in k[T] := k[T_1, \dots, T_r]$.

\Leftarrow : Assume that $\text{Im}(\alpha)$ is closed and that $\alpha: k^r \rightarrow \text{Im}(\alpha)$ is an isomorphism. Then there is a map $\beta = (g_1, \dots, g_r): k^n \rightarrow k^r$, with each polynomial g_i and element of $k[X_1, \dots, X_n]$, such that $\beta \circ \alpha = 1_{k^r}$. This means that $g_i(f_1(t), \dots, f_n(t)) = t$ for all $t \in k^r$ and all i . Because k is infinite, this implies that $T_i = g_i(f_1, \dots, f_n)$ for all $i \in \{1, \dots, r\}$. So α^* is surjective, which means that α is an embedding.

\Rightarrow : Suppose that α is an embedding, say $g_1, \dots, g_r \in k[X_1, \dots, X_n]$ are polynomials such that $g_i(f_1, \dots, f_n) = T_i$, for all i .

It follows from Lemma 5.1.2 below that $\text{Im}(\alpha) = V(\text{Ker } \alpha^*)$. So $\text{Im}(\alpha)$ is closed. Let $\beta: \text{Im}(\alpha) \rightarrow k^r$ be the regular map defined by $\beta := (g_1, \dots, g_r)$. Then obviously $\beta \circ \alpha(t) = t$ for all $t \in k^r$. Furthermore, if $x \in \text{Im}(\alpha) = V(\text{Ker } \alpha^*)$, then it also follows from Lemma 5.1.2 that $x_i = f_i(g_1(x), \dots, g_r(x))$ for all i , i.e., $x = \alpha \circ \beta(x)$. In other words, $\alpha: k^r \rightarrow \text{Im}(\alpha)$ is an isomorphism with inverse β . \square

Lemma 5.1.2. *Let $\alpha = (f_1, \dots, f_n): R \rightarrow R^n$ be an embedding with each $f_i \in R[T]$. Let $g_1, \dots, g_r \in R[X] := R[X_1, \dots, X_n]$ be polynomials such that $g_i(f_1, \dots, f_n) = T_i$ for all $i \in \{1, \dots, r\}$. Then*

$$\text{Ker}(\alpha^*) = (X_1 - f_1(g_1, \dots, g_r), \dots, X_n - f_n(g_1, \dots, g_r)).$$

Proof \subseteq : Let $p \in k[X]$ with $p(f_1, \dots, f_r) = 0$. Write

$$p = p(f_1, \dots, f_n) + \sum_{i=1}^n a_i(X, U)(X_i - f_i(U))$$

for some $a_i(X, U) \in k[X, U]$. Substituting $U_i := g_i$ for all i , we find that $p \in (X_1 - f_1(g_1, \dots, g_r), \dots, X_n - f_n(g_1, \dots, g_r))$, since no U_i appears in p .

\supseteq : Easy. \square

An embedding $\alpha := (f_1(T), \dots, f_n(T)): R^r \rightarrow R^m$ is called *rectifiable* if there is an invertible polynomial map $F: R^n \rightarrow R^n$ such that

$$F \circ \alpha = (T_1, \dots, T_r, 0, \dots, 0).$$

Equivalently, α is rectifiable if $\alpha^* \circ F^* = i^*$. Here $F^*: R[X] \rightarrow R[X]$ is the polynomial automorphism of $R[X]$ induced by F and $i^*: R[X] \rightarrow R[T]$ is the ring homomorphism induced by the inclusion $i: R^r \rightarrow R^n$.

Problem 5.1.3 (Embedding Problem). *Let k be an algebraically closed field of characteristic zero and let $\alpha: k^r \rightarrow k^n$ be an embedding. Is it then true that α is always rectifiable?*

The case $r = 1$ and $n = 2$ was answered affirmatively by Abhyankar and Moh in [AM75] and Suzuki in [Suz74]. A little later it was conjectured by Abhyankar in [Abh78] that for $n \geq 3$ there do exist embeddings of k in k^n which are not rectifiable. However, Craighero showed in [Cra86] that for $n \geq 4$ every embedding of k in k^n is rectifiable. The same result was obtained by Jelonek in [Jel87]. In fact Jelonek showed that if $n \geq 2r + 2$, then every embedding of k^r in k^n is rectifiable, while Craighero showed this for all $n \geq 3r + 1$. See also the paper [Sri91] of Srinivas for a generalisation of this result.

The case $r = 1$ and $n = 3$ remains open. A possible counterexample is discussed in Section 5.5. For more results about embeddings of k in k^3 , see the paper [BR91] of Bhatwadekar and Roy.

The following easy argument, due to Jelonek in [Jel87], shows that every embedding $\alpha := (f_1(T), \dots, f_n(T)): k^r \rightarrow k^n$ is *stably rectifiable*, i.e., there exists $m \geq 1$ such that $\tilde{\alpha}: k^r \rightarrow k^{n+m}$ defined by $\tilde{\alpha} := (f_1, \dots, f_n, 0, \dots, 0)$ is rectifiable.

Proposition 5.1.4. *Let $\alpha = (f_1(T), \dots, f_n(T)): R^r \rightarrow R^n$ be an embedding. Then $\tilde{\alpha} := (f_1(T), \dots, f_n(T), 0, \dots, 0): R^r \rightarrow R^{n+r}$ is rectifiable.*

Proof. Let $g = (g_1, \dots, g_r) \in R[X]^r$ be such that $g_i(f_1, \dots, f_n) = T_i$ for all i . Then both $G := (X, T_1 + g_1, \dots, T_r + g_r)$ and $H := (X_1 - f_1, \dots, X_n - f_n, T)$ are R -automorphisms of R^{n+r} . Take $F := H \circ G$. Then one easily verifies that $F \circ \tilde{\alpha} = (0, \dots, 0, T_1, \dots, T_r)$. \square

5.2 Coordinates and embeddings

The following lemma shows the relation between rectifiable embeddings and coordinates.

Lemma 5.2.1. *Let α be an embedding of R in R^n . Then α is rectifiable if and only if $\text{Ker}(\alpha^*)$ contains a coordinate.*

Proof The implication \Rightarrow is easy. So suppose that $\text{Ker}(\alpha^*)$ contains a coordinate. Let F be an invertible polynomial map from R^n to R^n which has this coordinate as its last component. Writing α again as (f_1, \dots, f_n) with each $f_i \in R[T]$, one gets that $F \circ \alpha$ is an embedding of the form $(f_1, \dots, f_{n-1}, 0)$. So, by Proposition 5.1.4, $F \circ \alpha$ is rectifiable. This implies that α itself is rectifiable. \square

In dimension two it is possible to use the Generalised Abhyankar-Moh-Suzuki Theorem (Theorem 4.1.5) to obtain the following.

Theorem 5.2.2. *Assume that R is a \mathbb{Q} -algebra and let $\alpha: R \rightarrow R^2$ be an embedding. Then α is rectifiable if and only if $\text{Ker}(\alpha^*)$ is a principal ideal.*

Proof \Rightarrow : Suppose that F is an invertible polynomial map from R^2 to R^2 rectifying α . So $F \circ \alpha = (T, 0)$. Since $\text{Ker}(F \circ \alpha)^*$ is a principal ideal, so is $\text{Ker}(\alpha^*)$.

\Leftarrow : Conversely, suppose that $\text{Ker}(\alpha^*) = (f)$ for some $f \in R[X, Y]$. Then $R[X, Y]/(f) \cong_R R[T]$ and hence, by Theorem 4.1.5, f is a coordinate. So by Lemma 5.2.1, α is rectifiable. \square

Now it is possible to give the main result of this section.

Theorem 5.2.3. *1. Assume that R is a \mathbb{Q} -algebra and assume that R is a unique factorisation domain. Then every embedding $\alpha: R \rightarrow R^2$ is rectifiable.*

2. Let $R := \mathbb{C}[Z^2, Z^3] \subseteq \mathbb{C}[Z]$. Take $f := T - Z^3T^2$ and $g := Z^2T$ in $R[T]$. Then $\alpha := (f, g)$ is a non-rectifiable embedding of R in R^2 .

Proof 1. Denote the quotient field $Q(R)$ of R by k . Note that α can also be considered as an embedding of k in k^2 .

By the Abhyankar-Moh-Suzuki Theorem, $\alpha: k \rightarrow k^2$ is rectifiable and therefore there exists a polynomial $f \in k[X, Y]$ such that

$$\text{Ker}(\alpha^*: k[X, Y] \rightarrow k[T]) = k[X, Y]f.$$

It is possible to choose f in such a way that $f \in R[X, Y]$ and such that f is primitive (over R). Then $\text{Ker}(\alpha^*: R[X, Y] \rightarrow R[T]) = R[X, Y] \cap k[X, Y]f = R[X, Y]f$. Now apply the previous theorem.

2. Take $F := X + Z^3X^2 - Z^3XY^3 + 2Y^3 - Z^2Y^5 \in R[X, Y]$. Then one easily verifies that $F(f, g) = T$. Hence α is an embedding.

To show that α is not rectifiable, it is enough to show that $\text{Ker}(\alpha^*)$ is not a principal ideal (Theorem 5.2.2). So assume that $\text{Ker}(\alpha^*) = (p)$ for some $p \in R[X, Y]$.

Take $b := Z^4X - Z^2Y + Z^3Y^2 \in R[X, Y]$. Then $b \in \text{Ker}(\alpha^*)$, so $b = ap$ for some $a \in R[X, Y]$. Looking at the coefficients of X and 1 in the expansions of b and ap , one easily deduces that $p = \lambda b$, for some $\lambda \in \mathbb{C}^*$. So, $\text{Ker}(\alpha^*) = (b)$.

Finally, $c := -Y + Z^2X + Z^3XY + Z^2Y^3 \in \text{Ker}(\alpha^*)$. Write $c = \tilde{a}b$ for some $\tilde{a} \in R[X, Y]$. Substituting $Z := 0$ and $X := 0$ in this equation gives $-1 = 0$, a contradiction. So α is not rectifiable. \square

The first part of this theorem was already obtained by Berson in [Ber99].

5.3 Derivations and embeddings

This section characterises embeddings in terms of locally nilpotent derivations. From here on, this chapter has been influenced very much by the paper [Asa99] of Asanuma (see Section 5.5).

Let $D = (D_1, \dots, D_r)$ be a sequence of pairwise commuting derivations on a ring A . A sequence $s = (s_1, \dots, s_r) \in A^r$ is called a *slice system* of D if $D_i(s_j) = \delta_{ij}$ for all i, j . If each D_i is locally nilpotent, then it follows easily from Theorem 1.1.2 that $A = A^D[s_1, \dots, s_r]$, a polynomial ring in s_1, \dots, s_r over $A^D := \cap_{i=1}^r A^{D_i}$.

Definition 5.3.1. Let $\alpha: R^r \rightarrow R^n$ be a polynomial map given by $\alpha := (f_1, \dots, f_n)$ for some polynomials $f_1, \dots, f_n \in R[U] := R[U_1, \dots, U_r]$. For each $i \in \{1, \dots, r\}$, define the triangular, hence locally nilpotent, derivation $D_{\alpha, i}$ by

$$D_{\alpha, i} := f_{1U_i} \partial_{X_1} + \dots + f_{nU_i} \partial_{X_n} + T \partial_{U_i}$$

on the $n + r + 1$ variable polynomial ring $R[T, U, X_1, \dots, X_n]$. One easily verifies that these derivations commute pairwise. Put $D_\alpha := (D_{\alpha, 1}, \dots, D_{\alpha, r})$.

Lemma 5.3.2. Assume that R is a \mathbb{Q} -algebra and let $f(U) \in R[U]$ be a polynomial in one variable. Then

$$\sum_{j=1}^{\infty} \frac{1}{j!} (-U)^j f^{(j)}(U) = f(0) - f(U).$$

Proof. Elementary: just take the derivative on both sides. \square

Lemma 5.3.3. *Assume that R is a \mathbb{Q} -domain. Let $f_1(U), \dots, f_n(U) \in R[U]$ and $b(T) \in R[T] \setminus \{0\}$ be polynomials in one variable over R . Let D be the locally nilpotent R -derivation on $R[T, U, X] := R[T, U, X_1, \dots, X_n]$ defined by*

$$D := f'_1(U)\partial_{X_1} + \dots + f'_n(U)\partial_{X_n} + b(T)\partial_U.$$

Let R_0 be the R -subalgebra of $R[T, U, X]$ generated by the elements $T, b(T)X_1 - f(U), \dots, b(T)X_n - f(U)$. Then $R_0 \subseteq R[T, U, X]^D \subseteq R_0[b(T)^{-1}]$.

Proof. (See also Section 1.4.) Note that U is a preslice of D and take $s := b(T)^{-1}U$. The derivation D can be extended to a derivation D on $R[T, U, X, b^{-1}]$ and by Theorem 1.1.3, $R[T, U, X, b^{-1}]^D = \varphi_{-s}(R[T, U, X, b^{-1}])$. Here φ_{-s} is as defined in Definition 1.1.1.

Now compute this algebra:

$$\begin{aligned} \varphi_{-s}(T) &= T - sD(T) + \dots = T, \\ \varphi_{-s}(U) &= U - sD(U) + \frac{1}{2}s^2D^2(U) + \dots \\ &= U - b(T)^{-1}Ub(T) = 0, \\ \varphi_{-s}(X_i) &= X_i + \sum_{j=1}^{\infty} \frac{1}{j!}(-s)^j D^j(X_i) \\ &= X_i + \sum_{j=1}^{\infty} \frac{1}{j!} \frac{(-U)^j}{b(T)^j} b(T)^{j-1} f_i^{(j)}(U) \\ &= X_i + \sum_{j=1}^{\infty} \frac{1}{j!} (-U)^j f_i^{(j)}(U) \\ &= X_i + \frac{1}{b(T)} (f_i(0) - f_i(U)) \quad (\text{by Lemma 5.3.2}), \\ \varphi_{-s}(b(T)^{-1}) &= b(T)^{-1} - sD(b(T)^{-1}) \\ &= b(T)^{-1}. \end{aligned}$$

Clearing the denominators of $\varphi_{-s}(X_i)$ by multiplying with $b(T)$, which is an element of the kernel, one finds that the R -algebra

$$\begin{aligned} R_0 &:= R[T, b(T)X_1 - f_1(U) + f_n(0), \dots, b(T)X_n - f_n(U) + f_n(0)] \\ &= R[T, b(T)X_1 - f_1(U), \dots, b(T)X_n - f_n(U)] \end{aligned}$$

satisfies $R_0 \subseteq R[T, U, X]^D \subseteq R_0[b(T)^{-1}]$. \square

Lemma 5.3.4. Assume that R is a \mathbb{Q} -domain. Let $\alpha = (f_1, \dots, f_n): R^r \rightarrow R^n$ be an embedding with each $f_i \in R[U] := R[U_1, \dots, U_r]$. Consider the sequence of derivations D_α on $R[T, U, X] := R[T, U_1, \dots, U_r, X_1, \dots, X_n]$. Then every element of A^{D_α} is equivalent modulo (T, X) to an element of $R[f_1, \dots, f_n]$.

Proof Let $i \in \{1, \dots, r\}$ and consider the derivation $D_{\alpha, i}$. By Lemma 5.3.3, the R -algebra

$$R_i := R[T, U_1, \dots, U_{i-1}, U_{i+1}, \dots, U_r, f_1 - TX_1, \dots, f_n - TX_n]$$

satisfies $R_i \subseteq R[T, U, X]^{D_{\alpha, i}} \subseteq R_i[T^{-1}]$. Taking the intersection over all $i \in \{1, \dots, r\}$, it follows that

$$R_0 \subseteq R[T, U, X]^{D_\alpha} \subseteq R_0[T^{-1}],$$

where $R_0 := R[T, f_1 - TX_1, \dots, f_n - TX_n]$.

Now let $q \in R[T, U, X]^{D_\alpha}$. Then, in particular, $q \in R_0[T^{-1}]$ so there is a $\rho \in \mathbb{N}$ and a polynomial p over R such that $T^\rho q = p(T, f_1 - TX_1, \dots, f_n - TX_n)$. Substituting $X_i := 0$ for all i and expanding the resulting right hand side in powers of T gives

$$T^\rho q = \sum_{i \geq 0} p_i(f_1, \dots, f_n) T^i$$

for some $p_i(f) \in R[f_1, \dots, f_n] \subseteq R[U]$. Since T^ρ divides the left hand side, it divides the right hand side as well. So $p_i(f) = 0$ for all $i < \rho$. Hence

$$q[X_1 := 0, \dots, X_n := 0] = \sum_{i \geq \rho} p_i(f_1, \dots, f_n) T^{i-\rho}.$$

Now substituting $T := 0$ one finds that $q \equiv p_0(f_1, \dots, f_n) \pmod{(T, X)}$. \square

Theorem 5.3.5. Assume that R is a \mathbb{Q} -domain and let $\alpha: R^r \rightarrow R^n$ be an embedding. The map α is an embedding if and only if the sequence of derivations D_α has a slice system.

Proof Write $\alpha = (f_1, \dots, f_n)$ with each $f_i \in R[U_1, \dots, U_r]$.

\Rightarrow : Assume that α is an embedding. Then each U_i can be written as $U_i = g_i(f_1, \dots, f_n)$ for some polynomial $g_i \in R[X]$. Define

$$s_i := \frac{U_i - g_i(f_1 - TX_1, \dots, f_n - TX_n)}{T}$$

for all $i \in \{1, \dots, r\}$. Since each $f_j - TX_j$ belongs to $R[T, U, X]^{D_\alpha}$ and $D_{\alpha, i}(U_j) = \delta_{ij}T$ for all i, j , it follows that (s_1, \dots, s_r) is a slice system of D_α .

\Leftarrow : Suppose that D_α has a slice system (s_1, \dots, s_r) in $R[T, U, X]^r$. Then, for all i , $D_{\alpha,i}(U_i - Ts_i) = 0$. Also $D_{\alpha,j}(U_i - Ts_i) = 0$ if $i \neq j$. So $U_i - Ts_i \in R[T, U, X]^{D_\alpha}$ for all i . Now use Lemma 5.3.4 and make the substitutions $T := 0$ and $X_i := 0$ for all i . It follows that each U_i is an element of $R[f_1, \dots, f_n]$. This means that α is an embedding. \square

Example 5.3.6. Let D be the derivation $D := (1 + Z^2)\partial_X + Z\partial_Y + T\partial_Z$ on the polynomial ring $\mathbb{C}[T, X, Y, Z]$. It was claimed in Counterexample 3.3.8 that D doesn't have a slice; this can now be shown.

The derivation is of the form $D = f'_1(Z)\partial_X + f'_2(Z)\partial_Y + T\partial_Z$ with $f_1(Z) := Z + \frac{1}{3}Z^3$ and $f_2(Z) := \frac{1}{2}Z^2$. Now one can easily see that $\mathbb{C}[f_1, f_2] \neq \mathbb{C}[Z]$, for instance by using the Abhyankar-Moh Theorem or by using Gröbner bases techniques. Hence, by the above theorem, D doesn't have a slice.

5.4 Rectifiability and the Cancellation Problem

Throughout this section, let k be some field of characteristic zero and let

$$\alpha = (f_1(U), \dots, f_n(U)): k^r \rightarrow k^n$$

denote an embedding. By the results of the previous section, one can associate to α a sequence $D_\alpha = (D_{\alpha,1}, \dots, D_{\alpha,r})$ of locally nilpotent derivations having a slice system on the $n + r + 1$ variable polynomial ring $k[T, U, X] := k[T, U_1, \dots, U_r, X_1, \dots, X_n]$.

In order to simplify the notations, write $f - TX$ instead of $f_1 - TX_1, \dots, f_n - TX_n$. The main result of this section, Theorem 5.4.1, asserts that if α is rectifiable, then the kernel of each derivation $D_{\alpha,i}$ is a polynomial ring in $n + r$ variables over k , which shows that the Cancellation Problem has an affirmative answer for these derivations.

Theorem 5.4.1. *Let $\alpha = (f_1(U), \dots, f_n(U)): k^r \rightarrow k^n$ be an embedding. Assume that α is rectifiable, say $G = (g_1(X), \dots, g_n(X)): k^n \rightarrow k^n$ is a polynomial automorphism such that $G \circ \alpha = (U_1, \dots, U_r, 0, \dots, 0)$.*

Take $i \in \{1, \dots, n\}$ and consider the derivation $D_{\alpha,i}$ on the $n + r + 1$ variable polynomial ring $k[T, U, X] := k[T, U_1, \dots, U_r, X_1, \dots, X_n]$. Then $k[T, U, X]^{D_{\alpha,i}}$ is generated, as a k -algebra, by the following $n + r$ elements:

- T ,
- U_j , for $j \in \{1, \dots, r\} \setminus \{i\}$,

- $s_j := T^{-1}(g_j(f - TX) - U_j)$, for $j \in \{1, \dots, r\} \setminus \{i\}$,
- $s_j := T^{-1}g_j(f - TX)$, for $j \in \{r+1, \dots, n\}$, and
- $s_i := g_i(f - TX)$.

Proof Let R be the ring $k[U_1, \dots, U_{i-1}, U_{i+1}, \dots, U_r]$ and consider $k[T, U, X]$ as a polynomial ring in T, U_i, X_1, \dots, X_n over R . Let R_0 be the R -subalgebra of $k[T, U, X]$ from Lemma 5.3.3. This means that R_0 is generated, as an R -algebra, by the following elements:

- $f_j - TX_j$ for $j \in \{1, \dots, n\}$.

Note that the k -algebra from the formulation of the theorem is, in fact, an R -subalgebra of $R[T, U_i, X]$ as well; denote it by R_1 .

Now I claim that

$$R_0 \subseteq R_1 \subseteq R[T, U_i, X]^{D_{\alpha, i}} \subseteq R_1[T^{-1}].$$

For the first of these inclusions, note that it is sufficient to show that each $g_j(f - TX)$, $j \in \{1, \dots, n\}$, is an element of R_1 , since one can then compose the inverse of G with the polynomial map $(g_1(f - TX), \dots, g_n(f - TX))$ to obtain that all $f_j - TX_j$, $j \in \{1, \dots, n\}$, are elements of R_1 . Now $g_i(f - TX) \in R_1$ by definition, for $j \in \{1, \dots, r\} \setminus \{i\}$ the fact that $g_j(f - TX) \in R_1$ follows readily from the fact that $g_j(f - TX) = Ts_j + U_j$, and for $j \in \{r+1, \dots, n\}$ it follows from the fact that $g_j(f - TX) = Ts_j$.

The second of these inclusions is easy; it is worth observing, though, that all generators of R_1 are indeed elements of $R[T, U_i, X]$ (and not just of the localisation $R[T, U_i, X, T^{-1}]$).

The last inclusion follows from the fact that $R_0 \subseteq R[T, U_i, X]^D \subseteq R_0[T^{-1}]$, which is Lemma 5.3.3, and the first two inclusions.

By Proposition 1.4.7, it is now enough to show that $R_1 : T = R_1$, in order to be able to conclude that $R[T, U_i, X]^{D_{\alpha, i}} = R_1$. Let $p(Y_0, Y_1, \dots, Y_n)$ be a polynomial over R in $n+1$ variables such that $\tilde{p} := p(T, s_1, \dots, s_n) \in R_1$ is divisible by T and let \tilde{q} be the quotient \tilde{p}/T . Note that, for all $j \in \{1, \dots, n\} \setminus \{i\}$,

$$s_j = g_{jX_1}(f)X_1 + \dots + g_{jX_n}(f)X_n + T(\dots)$$

and that $s_i = U_i + T(\dots)$. Divisibility of \tilde{p} by T therefore means that

$$\begin{aligned} p(0, g_{1X_1}(f)X_1 + \dots + g_{1X_n}(f)X_n, \dots, U_i, \dots \\ \dots g_{nX_n}(f)X_n + \dots + g_{1X_1}(f)X_n) = 0. \end{aligned} \quad (5.1)$$

Since $\det(JG) \in k^*$, also $\det(JG)(f) \in k^*$. Consequently, the X -linear forms $g_{jX_1}(f)X_1 + \cdots + g_{jX_n}(f)X_n$, $j \in \{1, \dots, n\}$ together with T and U_i form an invertible polynomial automorphism of $R[T, U_i, X]$. Therefore (5.1) implies that Y_0 divides p , say $q := p/Y_0$. Then $\bar{q} = q(T, s_1, \dots, s_n) \in R_1$.

So $R_1 : T = R_1$ and therefore $k[T, U, X]^{D_{\alpha,i}} = R[T, U_i, X]^{D_{\alpha,i}} = R_1$. \square

Corollary 5.4.2. *If α is rectifiable, then $A^{D_{\alpha,i}} \cong_{k[T]} k[T]^{[n+r-1]} \cong_k k^{[n+r]}$.* \square

As another consequence of Theorem 5.4.1, a new class of locally nilpotent derivations for which the Cancellation Problem has an affirmative answer can be described.

In order to do this, consider the $n + 2$ variable polynomial ring $k[T, U, X] := k[T, U, X_1, \dots, X_n]$ over k . So $r = 1$ in the notation of the previous theorem. Consider a derivation of the form

$$D = a_1(U)\partial_{X_1} + \cdots + a_n(U)\partial_{X_n} + b(T)\partial_U$$

with $b(T) \neq 0$. Of course one can write such a derivation as

$$D = f'_1(U)\partial_{X_1} + \cdots + f'_n(U)\partial_{X_n} + b(T)\partial_U$$

with $f_i(0) = 0$ for all i .

The remainder of this section will be dedicated to proving the following theorem.

Theorem 5.4.3. *Let $n \neq 3$. If $D = f'_1(U)\partial_{X_1} + \cdots + f'_n(U)\partial_{X_n} + b(T)\partial_U$ has a slice in $k[T, U, X]$, then $k[T, U, X]^D \cong_{k[T]} k[T]^{[n]} \cong_k k^{[n+1]}$.*

In the case that $\deg b = 0$, $s := b^{-1}U$ is a slice of D and the result immediately follows. So from now on, assume that $\deg b > 0$. To prove the theorem, the following two lemmas are used. Let π denote the substitution homomorphism defined by $\pi(g(T, U, X)) = g(b(T), U, X)$ for all $g \in k[T, U, X]$ and let D' be the derivation $D' := f'_1(U)\partial_{X_1} + \cdots + f'_n(U)\partial_{X_n} + T\partial_U$.

Lemma 5.4.4. *Let $N \geq 1$ and let $g_1(T, U, X), \dots, g_N(T, U, X) \in k[T, U, X]$. Let I be the ideal of all polynomials $p \in k[Y_0, Y] := k[Y_0, Y_1, \dots, Y_N]$ such that the polynomial $p(T, g_1(b(T), U, X), \dots, g_N(b(T), U, X))$ is divisible by $b(T)$ and let J be the ideal of all polynomials $p \in k[Y]$ such that $p(g_1, \dots, g_N)$ is divisible by T . Then $I = k[Y_0, Y]b(Y_0) + k[Y_0, Y]J$.*

Proof. \subseteq : Let $p \in I$. Write $p = \sum_{i=0}^{(\deg b)-1} p_i(Y)Y_0^i + b(Y_0)\tilde{p}(Y_0, Y)$ for some polynomials $p_i \in k[Y]$ and $\tilde{p} \in k[Y_0, Y]$. Then by definition

$$b(T) \mid \sum_{i=0}^{(\deg b)-1} p_i(g_1(0, U, X), \dots, g_N(0, U, X))T^i.$$

Since the T -degree of this sum is smaller than $\deg b$, it follows that

$$p_i(g_1(0, U, X), \dots, g_N(0, U, X)) = 0 \text{ for all } i.$$

So $p_i \in J$.

\supseteq : Obvious. □

The next lemma shows that the kernel algorithm of Section 1.4 does “basically the same” when computing the kernel of D and D' .

Lemma 5.4.5. *Let R_0 be the k -subalgebra of $k[T, U, X]$ generated by $T, b(T)X_1 - f_1(U), \dots, b(T)X_n - f_n(U)$ and let R'_0 be the k -subalgebra generated by $T, TX_1 - f_1(U), \dots, TX_n - f_n(U)$ (See also Lemma 5.3.3). Then, for every $i \in \mathbb{N}$, $R_0 : b(T)^i = \pi(R'_0 : T^i)[T]$.*

Proof. By induction on i .

$i = 0$: Trivial.

$i > 0$: Assume that

$$R_0 : b(T)^i = k[T, g_1(b(T), U, X), \dots, g_N(b(T), U, X)]$$

and that

$$R'_0 : T^i = k[T, g_1(T, U, X), \dots, g_N(T, U, X)]$$

for some $N \geq 1$ and some polynomials $g_j, j \in \{1, \dots, N\}$. The result now follows from the previous lemma. □

Corollary 5.4.6. *If D has a slice in $k[T, U, X]$, then α is an embedding.*

Proof. Take $s \in k[T, U, X]$ such that $D(s) = 1$. Then $D(U - b(T)s) = 0$, i.e., $U - b(T)s \in k[T, U, X]^D$. Since $k[T, U, X]^D = \cup_{i \geq 0} R_0 : b(T)^i$, it follows from Lemma 5.4.5 that

$$U - b(T)s = P(T, g_1(b(T), U, X), \dots, g_N(b(T), U, X))$$

for some $g_i(b(T), U, X) \in k[T, U, X]^D$ and some polynomial P over k .

Let c be a root of $b(T)$ in the algebraic closure \bar{k} of k . Substituting $T := c$ gives $U - p(c, g_1(0, U, X), \dots, g_N(0, U, X)) = 0$. Now, choosing a k -basis of \bar{k} containing 1, one deduces that there exists a $p \in k[Y_1, \dots, Y_N]$ such that $h := U - p(g_1(T, U, X), \dots, g_N(T, U, X))$ is divisible by T . Since, by Lemma 5.4.5, each $g_i(T, U, X)$ is an element of $k[T, U, X]^{D'}$, it follows that $s := T^{-1}h$ is a slice of D' in $k[T, U, X]$. So by Theorem 5.4.1 α is an embedding. \square

Proof of Theorem 5.4.3. By Corollary 5.4.6 α is an embedding. So, α is rectifiable (the case $n = 1$ is obvious, the case $n = 2$ is the Abhyankar-Moh-Suzuki Theorem, and if $n \geq 4$ one can apply Craighero's and Jelonek's Theorem). Since by Lemma 5.4.5 $k[T, U, X]^D = \pi(k[T, U, X]^{D_1})[T]$, the desired result follows immediately from the $r = 1$ case of Theorem 5.4.1. \square

5.5 Possible counterexamples to problems on affine space

Theorem 5.4.1 showed that if an embedding is rectifiable, the kernels of the corresponding derivations are polynomial rings. So, in order to find a possible counterexample to the Cancellation Problem, it seems natural to look for non-rectifiable embeddings. A class of candidates of such embeddings was constructed by Shastri in [Sha92].

More precisely, let $r = 1$ and $n = 3$. He showed that every (open) knot-type has a real polynomial representation which defines an embedding of \mathbb{C} in \mathbb{C}^3 . In particular, he obtained the following polynomial representation of the trefoil knot by putting $f(U) := U^3 - 3U$, $g(U) := U^4 - 4U^2$, and $h(U) := U^5 - 10U$ and $\alpha(u) := (f(u), g(u), h(u))$. Indeed this α gives an embedding of \mathbb{C} in \mathbb{C}^3 , since one easily verifies that $F(f(U), g(U), h(U)) = U$, with $F(X, Y, Z) := YZ - X^3 - 5XY + 2Z - 7X$. This embedding α will be called the *Shastri map*.

Since $\alpha: \mathbb{R} \rightarrow \mathbb{R}^3$ represents the trefoil, it is not rectifiable over \mathbb{R} . This led Shastri to conjecture that $\alpha: \mathbb{C} \rightarrow \mathbb{C}^3$ is not rectifiable over \mathbb{C} as well. So in light of Theorem 5.4.1, the following conjecture seems reasonable.

Conjecture 5.5.1. Let $D := f'(U)\partial_X + g'(U)\partial_Y + h'(U)\partial_Z + T\partial_U$ on the polynomial ring $\mathbb{C}[T, U, X, Y, Z]$. Then $\mathbb{C}[T, U, X, Y, Z]^D \not\cong_{\mathbb{C}} \mathbb{C}^{[4]}$.

Since D has a slice, namely $s := T^{-1}(U - F(f(U) - TX, g(U) - TY, h(U) - TZ))$, and $\mathbb{C}[T, U, X, Y, Z]^D \cong_{\mathbb{C}} \mathbb{C}[T, U, X, Y, Z]/(s)$, this conjecture is equivalent to the following conjecture.

Conjecture 5.5.2. $\mathbb{C}[T, U, X, Y, Z]/(s) \not\cong_{\mathbb{C}} \mathbb{C}^{[4]}$.

So if these conjectures are true, the Cancellation Problem would be answered negatively. Hence, by Proposition 1.6.2, the answer to the Linearisation Problem would also be negative. Also, by Theorem 5.4.1, it would show that Shastri's embedding is indeed a counterexample to the Embedding Problem.

A similar conjecture was made by Asanuma in [Asa99]. To relate his conjecture with Conjecture 5.5.1, some of Asanuma's results are briefly recalled here.

If I is an ideal in $R := k[X]$, then the *Rees ring* associated to I , denoted by $\mathcal{R}_R(I)$, is the $R[T]$ -subalgebra of $R[T, T^{-1}]$ generated by the elements $T^{-1}i$ with $i \in I$. Suppose now that $R/I \cong_k k^{[1]}$. In other words, suppose that $I = \text{Ker}(\alpha^*)$ for some embedding α of k in k^n . Then it was shown in [Asa99] that $\mathcal{R}_R(I)^{[1]} \cong_{k[T]} k[T]^{[n+1]} \cong_k k^{[n+2]}$.

Conjecture 5.5.3 (Asanuma's Conjecture). *Let $I := \text{Ker}(\alpha^*)$, where $\alpha: \mathbb{C} \rightarrow \mathbb{C}^3$ is the Shastri map. Then $\mathcal{R}_R(I) \not\cong_{\mathbb{C}} \mathbb{C}^{[4]}$.*

The equivalence between Asanuma's Conjecture and Conjecture 5.5.1 follows immediately from Corollary 5.5.5 below. In order to establish this equivalence, let $\alpha = (f_1, \dots, f_n): k \rightarrow k^n$ be any embedding and let $\alpha^*: k[X] \rightarrow k[U]$ be the induced homomorphism of coordinate rings. Put $I := \text{Ker}(\alpha^*)$, the ideal of relations between f_1, \dots, f_n over k . Let $I(f - TX)$ denote the ideal of $k[T, U, X]$ obtained from I by making the substitutions $X_i := f_i - TX_i$ for all i . Finally, let $D := D_\alpha$ denote the derivation $f'_1 \partial_{X_1} + \dots + f'_n \partial_{X_n} + T \partial_U$ on $k[T, U, X]$, corresponding to α .

Proposition 5.5.4. $k[T, U, X]^D = k[T, f_1 - TX_1, \dots, f_n - TX_n, T^{-1}I(f - TX)]$.

Proof. Let $\tilde{\alpha} := (f_1, \dots, f_n, 0): k \rightarrow k^{n+1}$. Let $\tilde{D} := D_{\tilde{\alpha}}$ be the derivation on the polynomial ring $k[T, U, X_1, \dots, X_{n+1}]$ corresponding to $\tilde{\alpha}$. Note that \tilde{D} is the extension of D to $k[T, U, X_1, \dots, X_{n+1}]$ by sending X_{n+1} to 0. So

$$k[T, U, X_1, \dots, X_{n+1}]^{\tilde{D}} = k[T, U, X_1, \dots, X_n]^D[X_{n+1}]. \quad (5.2)$$

By Proposition 5.1.4, $\tilde{\alpha}$ is rectifiable. Let $F = (F_1, \dots, F_{n+1})$ be a k -automorphism of k^{n+1} rectifying $\tilde{\alpha}$. Then one easily verifies that $\tilde{I} := \text{Ker}(\tilde{\alpha}^*)$ equals (F_2, \dots, F_{n+1}) in $k[X_1, \dots, X_{n+1}]$. So, by Theorem 5.4.1 applied to the case

$r = 1$, one finds

$$\begin{aligned}
k[T, U, X_1, \dots, X_{n+1}]^{\tilde{D}} &= \\
&= k[T, F_1(f - TX, -TX_{n+1}), T^{-1}F_2(f - TX, -TX_{n+1}), \dots, \\
&\quad \dots, T^{-1}F_{n+1}(f - TX, -TX_{n+1})] \\
&= k[T, F_1(f - TX, -TX_{n+1}), \dots, \\
&\quad \dots, F_{n+1}(f - TX, -TX_{n+1}), T^{-1}\tilde{I}(f - TX, -TX_{n+1})] \\
&= k[T, f_1 - TX_1, \dots, f_n - TX_n, -TX_{n+1}, T^{-1}\tilde{I}(f - TX, -TX_{n+1})] \\
&= k[T, f_1 - TX_1, \dots, f_n - TX_n, -TX_{n+1}, T^{-1}I(f - TX), X_{n+1}],
\end{aligned}$$

where $I := \text{Ker}(\alpha^*)$ (since $\tilde{I} = Ik[X, X_{n+1}] + X_{n+1}k[X, X_{n+1}]$). The desired result now follows using (5.2). \square

Corollary 5.5.5. $k[T, U, X]^D \cong_k \mathcal{R}_{k[X]}(I)$.

Proof. This follows readily from the previous proposition by sending the variable X_i to the polynomial $T^{-1}(f - X_i)$ for all i . \square

Chapter 6

Estimation of Degrees

This final chapter is about a rather different subject than the previous ones. It tries to estimate the degree of the inverse of a polynomial automorphism in terms of the degree of the automorphism itself and the number of variables.

This chapter is also rather unfinished, in the sense that it leaves open a lot of questions about these estimations. Hence, it may provide a good starting point for further research.

6.1 Inverse degrees and the Jacobian Conjecture

Definition 6.1.1. Let $F: R^n \rightarrow R^n$ be a polynomial map over a ring R . The *degree* of F , denoted by $\deg F$, is defined to be $\max\{\deg f_1, \dots, \deg f_n\}$.

It was already mentioned in Chapter 1 that if the Jacobian Conjecture is true for polynomial maps from \mathbb{C}^n to \mathbb{C}^n , then it is also true for polynomial maps from R^n to R^n , for an arbitrary \mathbb{Q} -algebra R .

Now, assume briefly that the Jacobian Conjecture is in fact true for such polynomial maps. Consider the “generic” polynomial map of degree d in n variables $X := X_1, \dots, X_n$:

$$F_u := (X_1 + \sum_{2 \leq |\alpha| \leq d} a_\alpha^{(1)} X^\alpha, \dots, X_n + \sum_{2 \leq |\alpha| \leq d} a_\alpha^{(n)} X^\alpha)$$

over the polynomial ring R_u over \mathbb{Q} in all variables $a_\alpha^{(i)} = a_{\alpha_1, \dots, \alpha_n}^{(i)}$ with $i \in \{1, \dots, n\}$, $\alpha \in \mathbb{N}^n$, $2 \leq |\alpha| := \alpha_1 + \dots + \alpha_n \leq d$. Let I be the ideal of R generated by the coefficients of $\det JF_u - 1$ and take R_u/I . Then F_u induces a polynomial map \bar{F}_u of degree d in n variables over R_u/I . Because $\det \bar{F}_u = 1$ and

the Jacobian Conjecture is assumed to be true, this polynomial map has an inverse, say \bar{G}_u .

Now, let $F = (f_1(X), \dots, f_n(X))$ be some (invertible) polynomial map of degree d in n variables over some \mathbb{Q} -algebra R with Jacobian determinant equal to 1. Composing F with $(X_1 - f_1(0), \dots, X_n - f_n(0))$ clears the constant part of F but doesn't change the degree of either F or F^{-1} . Similarly, composing F with the inverse of its linear part changes F into an invertible polynomial map of the form $(X_1 + \text{h.o.t.}, \dots, X_n + \text{h.o.t.})$, but doesn't change the degree of either F or F^{-1} . So as far as the degree of F or F^{-1} is concerned, it is safe to assume that F is of the form $(X_1 + \text{h.o.t.}, \dots, X_n + \text{h.o.t.})$. There is an obvious ring homomorphism from R_u to R sending $a_\alpha^{(i)}$ to the coefficient of X^α in f_i . This ring homomorphism induces a ring homomorphism from R_u/I to R , because $\det JF = 1$. Extending this ring homomorphism to $(R_u/I)[X]^n \rightarrow R[X]^n$ in the obvious way, the image of \bar{F}_u is exactly F and the image of \bar{G}_u is exactly F^{-1} . Therefore, the degree of F^{-1} is at most the degree of the “generic” inverse.

This suggests the following definition.

- Definition 6.1.2.** 1. Let R be a ring. Define $C(R, n, d) \in \mathbb{N} \cup \{\infty\}$ to be the supremum of $\deg F^{-1}$ over all polynomial automorphisms $F: R^n \rightarrow R^n$ with $\det JF = 1$.
2. Define $C(n, d) \in \mathbb{N} \cup \{\infty\}$ to be supremum of $C(R, n, d)$ over all \mathbb{Q} -algebras R .

Example 6.1.3. Fournie, Furter, and Pinchon have computed $C(2, 3)$; it equals 9 (see [FFP98]). Their method of computing this is to take the “generic” map F in two variables of degree 3 over the “universal ring” R_u and “guess” the degree of the inverse. Say G is the formal inverse of F up to degree 9. They then composed the original map F and its (guessed) inverse G , still over the universal ring R_u . Letting I be the ideal of R_u generated by the coefficients of $\det JF - 1$, all coefficients of $F \circ G$ turn out to be elements of I , showing that F and G are each other's inverse over the ring R_u/I . This is a very difficult computation because the composition $F \circ G$ is extremely large.

Another way to compute the inverse of the generic map F over R_u/I is by using Proposition 1.5.5. Letting D be the derivation $Y_1 \frac{\partial}{\partial F_1} + \dots + Y_n \frac{\partial}{\partial F_n}$ on the polynomial ring $R_u/I[X_1, \dots, X_n, Y_1, \dots, Y_n]$, it turns out that $D^{10} = 0$. See Appendix A for a computation with a computer algebra system.

The reasoning above actually shows the implication $2 \Rightarrow 1$ of the next theorem. This is one approach taken by Bass, Connell, and Wright in [BCW82] to attack the Jacobian Conjecture. The implication $1 \Rightarrow 2$ was proven by Bass in [Bas83].

Theorem 6.1.4. *For every $n \in \mathbb{N}^*$ and every $d \in \mathbb{N}^*$, the following two statements are equivalent:*

1. $C(n, d) < \infty$;
2. *the Jacobian Conjecture is true for all polynomial maps $F: \mathbb{C}^n \rightarrow \mathbb{C}^n$ with $\deg F \leq d$.* \square

Derksen has remarked in [Der94] that one does not have to consider all \mathbb{Q} -algebras in the definition of $C(n, d)$, but that one can restrict the attention to certain easier rings.

Definition 6.1.5. Define $C_\epsilon(n, d) \in \mathbb{N} \cup \{\infty\}$ to be supremum of $C(R, n, d)$ over all \mathbb{Q} -algebras R of the form $\mathbb{C}[T]/(T^m)$ for some $m \in \mathbb{N}^*$.

Theorem 6.1.6. *For every $n \in \mathbb{N}^*$ and every $d \in \mathbb{N}^*$, the following two statements are equivalent:*

1. $C_\epsilon(n, d) < \infty$;
2. *the Jacobian Conjecture is true for all polynomial maps $F: \mathbb{C}^n \rightarrow \mathbb{C}^n$ with $\deg F \leq d$.* \square

The reason for restricting the attention to polynomial maps with Jacobian determinant 1 is that, in general, there is no a priori bound on the degree of the inverse of a polynomial automorphism solely in terms of the number of variables and the degree of the automorphism itself. The following example, taken from [Ess00], page 56, shows this.

Example 6.1.7. Take $R := R[T]/(T^m)$ and write $\epsilon := \bar{T}$, so $R = k[\epsilon]$. Let $F: R \rightarrow R$ be the polynomial map $(X - \epsilon X^2)$. Since $\det JF = 1 - 2\epsilon X \in R[X]^*$ (and since the Jacobian Conjecture is true in 1 variable), this map is invertible, say $G: R \rightarrow R$ is its inverse. Then $G(F) = X$ and taking derivatives one sees that

$$1 = G'(F) \cdot F' = G'(X - \epsilon X^2) \cdot (1 - 2\epsilon X).$$

Therefore

$$G'(X - \epsilon X^2) = 1 + (2\epsilon X) + (2\epsilon X)^2 + \cdots + (2\epsilon X)^{m-1}.$$

Now, look at the degrees involved. It follows that

$$2 \deg G > 2 \deg G' \geq \deg G'(X - \epsilon X^2) = m - 1$$

and therefore $\deg G > \frac{m-1}{2}$. So the degree of the inverse of F can be arbitrary large.

The reason that the degree of the inverse becomes so large in the above example is that the Jacobian determinant of F is not a unit in R , but in $R[X]$. The (multiplicative) inverse of this Jacobian determinant has a large degree, causing F^{-1} to have a large degree as well.

If the ring is known to be a field of characteristic zero, then one can estimate the degree of the inverse in terms of the number of variables and the degree of the original invertible map. The following theorem appears in the paper [BCW82] and in [RW84]. Together with the next example, it shows that over a field of characteristic zero everything is already known.

Theorem 6.1.8. *Let $F: k^n \rightarrow k^n$ be a polynomial automorphism. Say $d := \deg F$. Then $\deg F^{-1} \leq d^{n-1}$.* \square

Example 6.1.9. Consider the polynomial map $F = (f_1, \dots, f_n): k^n \rightarrow k^n$ given by

$$F = (X_1 - X_2^d, X_2 - X_3^d, \dots, X_{n-1} - X_n^d, X_n).$$

Then

$$\begin{aligned} X_n &= f_n, \\ X_{n-1} &= f_{n-1} + X_n^d \\ &= f_{n-1} + f_n^d, \\ X_{n-2} &= f_{n-2} + X_{n-1}^d = f_{n-2} + (f_{n-1} + f_n^d)^d \\ &= f_{n-2} + f_{n-1}^d + \dots + f_n^{d^2}, \\ X_{n-3} &= f_{n-3} + X_{n-2}^d = f_{n-3} + (f_{n-2} + f_{n-1}^d + \dots + f_n^{d^2})^d \\ &= f_{n-3} + f_{n-2}^d + \dots + f_n^{d^3}, \\ &\vdots \\ X_1 &= f_1 + X_2^d = f_1 + (f_2 + f_3^d + \dots + f_n^{d^{n-2}})^d \\ &= f_1 + f_2^d + \dots + f_n^{d^{n-1}}. \end{aligned}$$

So F is invertible and $\deg F^{-1} = d^{n-1}$.

Note that the polynomial map F in the above example is triangular and the maximal degree of an inverse can therefore be obtained by a triangular map. The next three sections also study the degree of the inverse of a triangular polynomial map. It turns out that for a triangular map over an arbitrary \mathbb{Q} -algebra one can

find an a priori bound on the degree of the inverse. In the case of 2 variables, a bound that was found by Furter is improved and it is shown that over an arbitrary \mathbb{Q} -algebra one cannot obtain the maximal degree of an inverse by means of a triangular map. In the last section an interesting class of invertible polynomial maps in two variables is constructed whose inverse have a relatively large degree. Nevertheless, it is concluded that there is (still) no a priori bound one could hope to obtain.

6.2 Triangular maps

The following theorem is well-known for the case that R is a field and can easily be proven by induction on n .

Theorem 6.2.1. *Let $F = (f_1(X_1), f_2(X_1, X_2), \dots, f_n(X_1, \dots, X_n)) : R^n \rightarrow R^n$ be a triangular polynomial map with $\det JF = 1$. Then F is invertible.*

Proof Consider a prime ideal \mathfrak{p} of R . Over the residue field $Q(R/\mathfrak{p})$ of R , F is invertible by the field case. Therefore, by Property 3.5.7, F itself is invertible. \square

For triangular maps there does exist a bound on the degree of an inverse.

Theorem 6.2.2. *Given $n \in \mathbb{N}^*$ and $d \in \mathbb{N}^*$, there exists a constant $C_\Delta(n, d) \in \mathbb{N}^*$ such that for every triangular polynomial map $F : R^n \rightarrow R^n$ with $\det JF = 1$ of degree d , the degree of the inverse of F is at most $C_\Delta(n, d)$.*

Proof Consider the polynomial map

$$F := (X_1 + a_2^{(1)}X^2 + \dots + a_d^{(1)}X^d, \dots, X_n + \sum_{\substack{\alpha \in \mathbb{N}^n \\ 2 \leq |\alpha| \leq d}} a_\alpha^{(n)}X^\alpha)$$

over the polynomial ring R_u over \mathbb{Q} in all variables $a_\alpha^{(i)}$, with $i \in \{1, \dots, n\}$ and $\alpha \in \mathbb{N}^i$, $2 \leq |\alpha| := \alpha_1 + \dots + \alpha_i \leq d$. Analogous to the previous section, let I be the ideal of R generated by the coefficients of $\det JF - 1$ and let \bar{F}_u be the polynomial map over R_u/I induced by F . Then $\det J\bar{F}_u = 1$ and hence \bar{F}_u is invertible by Theorem 6.2.1. Let $C_\Delta(n, d)$ be the degree of \bar{F}_u^{-1} . With exactly the same argument as in the previous section, one sees that this is the required bound. \square

6.3 One variable

This section proves the following two theorems. They will be used in the next section to give an estimate for the degree of the inverse of an invertible triangular polynomial map in two variables.

Theorem 6.3.1. *Let R be a ring and $d \in \mathbb{N}$. Let $a_1, \dots, a_d \in R$ and suppose that the polynomial $f := 1 + a_1X + \dots + a_dX^d$ is an invertible element of the polynomial ring $R[X]$. Let $e \in \mathbb{N}$ be the degree of its inverse. Then, for every $\alpha_1, \dots, \alpha_d \in \mathbb{N}$ with $\alpha_1 + 2\alpha_2 + \dots + d\alpha_d \geq de + 1$, $a_1^{\alpha_1}a_2^{\alpha_2} \dots a_d^{\alpha_d} = 0$.*

Theorem 6.3.2. *Let R be a \mathbb{Q} -algebra and $d \in \mathbb{N}^*$. Let $a_1, \dots, a_d \in R$ and consider the polynomial $F := X + a_1X^2 + \dots + a_{d-1}X^d \in R[X]$. Assume that F' is invertible in $R[X]$ and let $e \in \mathbb{N}$ be the degree of its inverse. Then F is invertible when considered as an R -endomorphism of $R[X]$ and the degree of its inverse is at most $(d-1)e + 1$.*

For every $d \in \mathbb{N}$, let $R^{(d)}$ be the polynomial ring

$$R^{(d)} := \mathbb{Z}[b_1, \dots, b_d].$$

Let $f^{(d)}$ be the polynomial

$$f^{(d)} := 1 + b_1X + \dots + b_dX^d \in R^{(d)}[X].$$

Considered as an element of $R^{(d)}[[X]]$, this polynomial is invertible. Denote the coefficient of X^i in the inverse of $f^{(d)}$ by $c_i^{(d)}$. So

$$\frac{1}{f^{(d)}} = 1 + c_1^{(d)}X + c_2^{(d)}X^2 + \dots \in R^{(d)}[[X]],$$

with each $c_i^{(d)} \in R^{(d)}$. Also define $c_0^{(d)} := 1$ and $c_{-i}^{(d)} := 0$ for all $i \in \mathbb{N}^*$. For every $e \in \mathbb{N}$, let $I_e^{(d)}$ be the ideal of $R^{(d)}$ given by

$$I_e^{(d)} := (c_{e+1}^{(d)}, c_{e+2}^{(d)}, \dots).$$

Remark 6.3.3. The polynomial $f^{(d)}$ is invertible over the quotient ring $R^{(d)}/I_e^{(d)}$ and its inverse has degree e . Furthermore, if R is some ring and f is some invertible polynomial over R with constant term equal to 1 and whose inverse has degree e , then there is a uniquely determined natural map from $R^{(d)}/I_e^{(d)}$ to R mapping the coefficients of $f^{(d)}$ to those of f . So the quotient ring $R^{(d)}/I_e^{(d)}$ is the universal ring for the situation in Theorem 6.3.1.

Lemma 6.3.4. For all $d \in \mathbb{N}$ and for all $i \in \mathbb{N}^*$,

$$c_i^{(d)} + b_1 c_{i-1}^{(d)} + b_2 c_{i-2}^{(d)} + \cdots + b_d c_{i-d}^{(d)} = 0.$$

Proof. Look at the coefficient of X^i in

$$\begin{aligned} 1 &= f^{(d)} \frac{1}{f^{(d)}} \\ &= (1 + b_1 X + \cdots + b_d X^d) \cdot \\ &\quad (\cdots + c_{-1}^{(d)} X^{-1} + c_0^{(d)} X^0 + c_1^{(d)} X^1 + c_2^{(d)} X^2 + \cdots). \end{aligned}$$

On the left hand side, this coefficient equals 0; on the right hand side, it equals $c_i^{(d)} + b_1 c_{i-1}^{(d)} + b_2 c_{i-2}^{(d)} + \cdots + b_d c_{i-d}^{(d)}$. \square

On the ring $R^{(d)}$, consider the grading ω that gives each b_i degree i . The additive subgroup of $R^{(d)}$ consisting of all ω -homogeneous elements of ω -degree i is denoted by $R_{(i)}^{(d)}$. So $R^{(d)} = \bigoplus_{i \in \mathbb{N}} R_{(i)}^{(d)}$. Lemma 6.3.4 implies that each $c_i^{(d)}$ is ω -homogeneous of ω -degree i . As a consequence, each ideal $I_e^{(d)}$ is ω -homogeneous. So $I_e^{(d)} = \bigoplus_{i \in \mathbb{N}} (I_e^{(d)} \cap R_{(i)}^{(d)})$.

Lemma 6.3.5. For all $d \in \mathbb{N}$, $I_0^{(d)} = (b_1, \dots, b_d)$.

Proof. By induction on i , it follows easily from Lemma 6.3.4 that $b_i \in I_0^{(d)}$, for $i = 1, \dots, d$. Therefore $(b_1, \dots, b_d) \subseteq I_0^{(d)}$. Since the generators of $I_0^{(d)}$ are all ω -homogeneous of ω -degree at least 1, $I_0^{(d)}$ does not contain a non-zero constant. Hence $I_0^{(d)} = (b_1, \dots, b_d)$. \square

Lemma 6.3.6. For all $d, e \in \mathbb{N}^*$, $b_d I_{e-1}^{(d)} \subseteq I_e^{(d)}$.

Proof. Look at the generators $c_e^{(d)}, c_{e+1}^{(d)}, \dots$ of $I_{e-1}^{(d)}$. Except for $c_e^{(d)}$, they are all elements of $I_e^{(d)}$: they are even generators. In particular, $b_d c_{e+i}^{(d)} \in I_e^{(d)}$ for all $i \in \mathbb{N}^*$. Furthermore, by Lemma 6.3.4,

$$b_d c_e^{(d)} = -c_{d+e}^{(d)} - b_1 c_{d+e-1}^{(d)} - \cdots - b_{d-1} c_{e+1}^{(d)} \in I_e^{(d)}.$$

So $b_d I_{e-1}^{(d)} \subseteq I_e^{(d)}$. \square

Lemma 6.3.7. For all $d \in \mathbb{N}^*$ and all $e \in \mathbb{N}$,

$$I_e^{(d-1)} R^{(d)} \subseteq I_e^{(d)} + b_d R^{(d)}.$$

Looking in ω -degree m only, this means that

$$I_e^{(d-1)} R^{(d)} \cap R_{(m)}^{(d)} \subseteq (I_e^{(d)} \cap R_{(m)}^{(d)}) + b_d R_{(m-d)}^{(d)} R^{(d)},$$

for all $m \in \mathbb{N}$, $m \geq d$.

Proof. By induction on i follows easily from Lemma 6.3.4 that

$$c_i^{(d)} - c_i^{(d-1)} \in b_d R^{(d)}$$

for all $i \in \mathbb{N}$. This then implies the lemma by looking at the generators of $I_e^{(d-1)}$. The second claim now follows from the fact the $I_e^{(d-1)}$ and $I_e^{(d)}$ are ω -homogeneous ideals. \square

Remark 6.3.8. For $i = 0, \dots, d-1$ it is even true that $c_i^{(d)} = c_i^{(d-1)}$, but that is unimportant here.

Proposition 6.3.9. For all $d, e \in \mathbb{N}$ and $m \in \mathbb{N} \geq de + 1$, $R_{(m)}^{(d)} \subseteq I_e^{(d)}$.

Proof. By induction on $d + e$. If $d = 0$, the claim is trivial: there are no ω -homogeneous polynomials of ω -degree at least 1 in $R^{(0)}$. If $e = 0$, then Lemma 6.3.5 implies the claim. So it is possible to assume that $d, e \geq 1$. It is also sufficient to consider a monomial $b_1^{\alpha_1} b_2^{\alpha_2} \dots b_d^{\alpha_d}$ of ω -degree $m \geq de + 1$.

Case 1: $\alpha_d \geq 1$. Suppose that $\alpha_d \geq 1$. By the induction hypothesis, it follows that $b_1^{\alpha_1} b_2^{\alpha_2} \dots b_d^{\alpha_d-1} \in I_{e-1}^{(d)}$, since this monomial has ω -degree $m - d \geq de + 1 - d = d(e-1) + 1$. Now, by Lemma 6.3.6, $b_1^{\alpha_1} b_2^{\alpha_2} \dots b_d^{\alpha_d} \in I_e^{(d)}$. Note that this means that $b_d R_{(m-d)}^{(d)} R^{(d)} \subseteq I_e^{(d)}$.

Case 2: $\alpha_d = 0$. Suppose that $\alpha_d = 0$. Because $b_1^{\alpha_1} b_2^{\alpha_2} \dots b_{d-1}^{\alpha_{d-1}}$ has ω -degree $m \geq de + 1 \geq (d-1)e + 1$ and because it is an element of $R^{(d-1)}$, it follows by the induction hypothesis that $b_1^{\alpha_1} b_2^{\alpha_2} \dots b_{d-1}^{\alpha_{d-1}} \in I_e^{(d-1)}$. Hence, by Lemma 6.3.7, this monomial is an element of $I_e^{(d)} + b_d R_{(m-d)}^{(d)} R^{(d)}$. By case 1, it is even an element of $I_e^{(d)}$. \square

Proof of Theorem 6.3.1. Let $\varphi: R^{(d)} \rightarrow R$ be the ring homomorphism sending each variable b_i to the coefficient a_i . Because the inverse of f has (normal) degree e , each element of $I_e^{(d)}$ is mapped to 0 by φ . Hence, by Proposition 6.3.9, each monomial of ω -degree at least $de + 1$ is mapped to 0 by φ . An element of R of the form $a_1^{\alpha_1} a_2^{\alpha_2} \dots a_d^{\alpha_d}$ with $\alpha_1 + 2\alpha_2 + \dots + d\alpha_d \geq de + 1$ is, however, exactly the image of such a monomial under φ . \square

Let $F^{(d)}$ be the polynomial

$$F^{(d)} := X + b_1 X^2 + b_2 X^3 + \dots + b_d X^{d+1} \in R^{(d)}[X].$$

This polynomial can be viewed as an element of $\text{End}_R^{(d)}(R^{(d)}[[X]])$ and as such it is invertible. Denote its inverse by $G^{(d)}$ and write

$$G^{(d)} = X + p_1^{(d)} X^2 + p_2^{(d)} X^3 + \dots \in R^{(d)}[[X]]$$

with each $p_i \in R^{(d)}$. Extend the grading ω on $R^{(d)}$ to $R^{(d)}[X]$ and $R^{(d)}[[X]]$ by giving X ω -degree -1 . Then $F^{(d)}$ is ω -homogeneous of ω -degree -1 . Because $F^{(d)}(G^{(d)}) = X$, $G^{(d)}$ is ω -homogeneous of ω -degree -1 as well. This means that each $p_i^{(d)}$ is ω -homogeneous of ω -degree i .

Proof of Theorem 6.3.2. The derivative $F' = 1 + 2a_1X + 3a_2X^2 + \cdots + da_{d-1}X^{d-1}$ of F is an invertible element of $R[X]$ whose inverse has degree e . Hence, by Theorem 6.3.1,

$$(2a_1)^{\alpha_1}(3a_2)^{\alpha_2} \cdots (da_{d-1})^{\alpha_{d-1}} = 0$$

whenever $\alpha_1 + 2\alpha_2 + \cdots + (d-1)\alpha_{d-1} \geq (d-1)e + 1$. Because R is a \mathbb{Q} -algebra, this means that in that case,

$$a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_{d-1}^{\alpha_{d-1}} = 0.$$

Now, let $\varphi: R^{(d-1)} \rightarrow R$ be the ring homomorphism sending each variable b_i to a_i . Denote the inverse of F by G . A priori this is an element of $R[[X]]$. Since the coefficient of X^i of G is the image under φ of $p_{i-1}^{(d-1)} \in R^{(d-1)}$, which is a ω -homogeneous polynomial of ω -degree $i-1$, such a coefficient vanishes if $i-1 \geq (d-1)e + 1$. So G is in fact an element of $R[X]$ and its degree is at most $(d-1)e + 1$. \square

6.4 Triangular maps in two variables

This section proves the following theorem.

Theorem 6.4.1. *Let R be a \mathbb{Q} -algebra and let $F = (f_1(X), f_2(X, Y)): R^2 \rightarrow R^2$ be an invertible, triangular polynomial map with $\det JF = 1$. Say $d := \deg F$. Then $\deg F^{-1} \leq d^2 - d + 1$.*

This improves upon a result from Furter, who proved in [Fur98] that $\deg F^{-1} \leq 4d^4$.

Proving this theorem requires some preparations. Using notations from the formulation of the theorem, consider once again the polynomial ring $R^{(d-1)} := \mathbb{Z}[b_1, \dots, b_{d-1}]$ from the previous section. Let $\varphi: R^{(d-1)} \rightarrow R$ be the ring homomorphism sending each variable b_i to the coefficient of X^{i+1} in the polynomial f_1 , just as in the proof of Theorem 6.3.2. Extend this map to a ring homomorphism $\varphi: R^{(d-1)}[X] \rightarrow R[X]$ by sending X to X .

Also consider once again the grading ω on $R[X]$ giving each variable b_i degree i and X degree -1 . Let $R_{(i)}^{(d-1)}$ denote the collection of elements of R that are ω -homogeneous of ω -degree i and let $R^{(d-1)}[X]_{(i)}$ denote the collection of elements of $R[X]$ that are ω -homogeneous of ω -degree i .

It is convenient to say that an element of $R[X]$ is ω -homogeneous of ω -degree i if it is the image under φ of an element of $R^{(d-1)}[X]_{(i)}$. Note, however, that $R[X]$ is not a graded ring and that, for instance, an element might be ω -homogeneous of two different ω -degrees. The important observation is that Theorem 6.3.1 claims that every ω -homogeneous element of R of large ω -degree is, in fact, 0. It is also convenient to let I_i denote the ideal of R generated by the ω -homogeneous elements of R of ω -degree at least i .

Proof of Theorem 6.4.1. It is possible to assume that $f_1(X) = X + p_1(X)$ and $f_2(X, Y) = Y + p_2(X, Y)$. Here $p_1(X) \in R[X]$ and $p_2(X, Y) \in R[X, Y]$ consist of monomials of degree at least 2.

Now, $1 = \det JF = \frac{\partial f_1}{\partial X} \cdot \frac{\partial f_2}{\partial Y} = f'_1(X) \frac{\partial f_2}{\partial Y}$. Looking at the term of lowest Y -degree in $\partial f_2 / \partial Y$, it follows that $\deg_Y f_2 = 1$. Say $f_2(X, Y) = q_1(X)Y + q_2(X)$ and note that $q_1(X) = \partial f_2 / \partial Y = f'_1(X)^{-1}$. So f_2 is of the form $f'_1(X)^{-1}Y + q_2(X)$.

Let $G = (g_1(X), g_2(X, Y))$ be the inverse of F . Just as f_2 must be linear in Y , so must g_2 be. Say $g_2(X, Y) = r_1(X)Y + r_2(X)$ and, just as with f_2 , $r_1(X) = g_1(X)^{-1}$. Because $F \circ G = (X, Y)$, it holds in particular that $f_1(g_1(X)) = X$ and therefore $f'_1(g_1(X)) \cdot g'_1(X) = 1$. So g_2 is of the form $f'_1(g_1(X))Y + r_2(X)$.

Take $e := \deg 1/f'_1(X)$ and note that $e \leq d-1$. Therefore, by Theorem 6.3.2, $\deg g_1 \leq (d-1)e + 1 \leq d^2 - 2d + 2$.

Note that f'_1 is ω -homogeneous of ω -degree 0 (i.e., $f'_1 \in \varphi(R^{(d-1)}[X]_{(0)})$) and g_1 is ω -homogeneous of ω -degree -1 . Therefore, since ω gives degree -1 to X , $f'_1(g_1(X))$ is ω -homogeneous of ω -degree 0 as well. So the coefficient of X^i in $f'_1(g_1(X))$ is ω -homogeneous of ω -degree i . By Theorem 6.3.1, such a coefficient is 0 when $i \geq (d-1)e + 1$, so the X -degree of $f'_1(g_1(X))$ is at most $(d-1)e \leq d^2 - 2d + 1$. The degree of $f'_1(g_1(X))Y$ is therefore at most $d^2 - 2d + 2$.

For the second part of g_2 , use

$$\begin{aligned} Y &= f_2(g_1(X), g_2(X, Y)) \\ &= f'_1(g_1(X))^{-1} \cdot g_2(X, Y) + q_2(g_1(X)) \\ &= Y + f'_1(g_1(X))^{-1}r_2(X) + q_2(g_1(X)). \end{aligned}$$

Apparently, $r_2(X) = -f'_1(g_1(X)) \cdot q_2(g_1(X)) = (-f'_1 \cdot q_2)[X := g_1(X)]$.

Because $\deg q_2 \leq \deg f_2 \leq d$, $\deg f'_1 \leq d-1$, and f'_1 is ω -homogeneous of ω -degree 0, $-f'_1 q_2$ is of the form

$$\begin{aligned} -f'_1 q_2 &= - \sum_{i=0}^{2d-1} \left(\sum_{j=0}^i (\text{coeff. of } X^j \text{ in } f'_1) \cdot (\text{coeff. of } X^{i-j} \text{ in } q_2) \right) X^i \\ &= c_0 + c_1 X + \cdots + c_d X^d + c_{d+1} X^{d+1} + \cdots + c_{2d-1} X^{2d-1} \end{aligned}$$

with $c_0, \dots, c_d \in R$ and for every $i \in \{1, \dots, d-1\}$, $c_{d+i} \in I_i$. Also write

$$r_2 = d_0 + d_1 X + \dots$$

with $d_0, d_1, \dots \in R$. Substituting $X := g_1(X)$ in $-f'_1 q_2$ shows that

$$d_i = \sum_{j=0}^{2d-1} c_j \cdot (\text{the coefficient of } X^i \text{ in } g_1(X)^j).$$

Since g_1 is ω -homogeneous of ω -degree -1 , g_1^j is ω -homogeneous of ω -degree $-j$ and hence the coefficient of X^i in g_1^j is ω -homogeneous of ω -degree $i-j$. So the element

$$c_j \cdot (\text{the coefficient of } X^i \text{ in } g_1(X)^j)$$

is an element of I_{i-j} if $j \leq d$ and of $I_{j-d+i-j} = I_{i-d}$ if $i > d$. In any case, it is an element of I_{i-d} . So d_i is an element of I_{i-d} . If $i-d \geq (d-1)e+1 = d^2-2d+2$, then such an element is always zero, by Theorem 6.3.1. Therefore $\deg r_2 \leq d^2-d+1$. \square

6.5 Examples in two variables

Let d be an integer, $d \geq 3$, and $m := d(d-1)+1$. Consider the ring $R := \mathbb{C}[T]/(T^m)$ and denote by ϵ the class of T in this ring. Over R , consider the polynomial ring $R[X, Y] = \mathbb{C}[\epsilon][X, Y]$. This section exhibits a polynomial map $H \in \text{Aut}_R(R[X, Y])$ of degree d with $\det(JH) = 1$ whose inverse probably has degree d^2 .

In order to construct such a polynomial map H , first define the polynomial map F by

$$F := \begin{pmatrix} X - \epsilon XY + \frac{1}{d} Y^d \\ Y + \frac{1}{2} \epsilon Y^2 + \cdots + \frac{1}{d-1} \epsilon^{d-2} Y^{d-1} - \epsilon^{d-1} X \end{pmatrix}.$$

Because $F \equiv (X, Y) \pmod{\epsilon}$, F is invertible. Note that $\det(JF) = 1 - \epsilon^d X$.

Let $g(X) \in R[X]$ be defined by

$$g(X) := \frac{1 - \sqrt{1 - 2\epsilon^d X}}{\epsilon^d}$$

i.e., formally take the Taylor expansion of $(1 - \sqrt{1 - 2\epsilon^d X})/\epsilon^d$ at $X = 0$. Because $\epsilon^m = 0$, this turns out to be a polynomial. Let G be the polynomial map $G := (g(X), Y)$. Note that G is invertible and that $G^{-1} = (X - \frac{1}{2}\epsilon^d X^2, Y)$.

Now, take $H := F \circ G$. Then H is invertible as well and

$$\begin{aligned} \det(JH) &= \det(JF)|_{(X:=g(X), Y:=Y)} \det(JG) \\ &= (1 - \epsilon^d g(X)) \frac{1}{\sqrt{1 - 2\epsilon^d X}} \\ &= 1. \end{aligned}$$

This particular construction was suggested by Derksen and is based on an earlier idea of Van den Essen.

On $R[X, Y]$ one can define a grading ω by letting ϵ have degree -1 , X degree d , and Y degree 1 . Then F is ω -homogeneous of degree $(d, 1)$. Also, G and H are ω -homogeneous of degree $(d, 1)$. One can easily see that the (normal) degree of H is $(d, d - 1)$.

Inverse

In order to compute the inverse of H , consider the derivations $D_1 := H_{2Y}\partial_X - H_{2X}\partial_Y$ and $D_2 := -H_{1Y}\partial_X + H_{1X}\partial_Y$. Then the coefficient of $X^i Y^j$ in the first component of H^{-1} is given by $\frac{1}{i!j!} D_1^i D_2^j(X)|_{X,Y:=0}$ and in the second component by $\frac{1}{i!j!} D_1^i D_2^j(Y)|_{X,Y:=0}$ (see Proposition 1.5.6).

Proposition 6.5.1. *The degree of the first component of H^{-1} is $\leq d^2$ and of the second component $\leq d^2 - d + 1$. Also, $D_2^{d^2}(X) \in R$.*

Proof. D_1 and D_2 are ω -homogeneous of degree -1 . Hence $D_1^i D_2^j(X)$ is ω -homogeneous of degree $d - i - j$, but this requires a factor of at least ϵ^m in front of every monomial if $i + j > d^2$. If $i + j = d^2$, then a factor of at least ϵ^m must appear in front of every non-constant monomial in $D_1^i D_2^j(X)$. This implies that $D_2^{d^2}(X) \in R$.

Similarly, $D_1^i D_2^j(Y)$ is ω -homogeneous of degree $1 - i - j$, but this requires a factor of at least ϵ^m in front of every monomial if $i + j > d^2 - d + 1 (= m)$. \square

Conjecture 6.5.2. *The degree of the first component of H^{-1} equals d^2 . More precisely, the coefficient of Y^{d^2} in the first component of H^{-1} does not equal 0. In other words, $D_2^{d^2}(X) \neq 0$. \square*

Computations

Brute force computations done by Hubbers and the author show that the conjecture is true for small values of d ; for $d \leq 17$ it has been checked. Also, if d is an odd prime, brute force computations suggest that $D_2^{d^2}(X) \equiv \epsilon^{d^2-d} \pmod{d}$. This can also be checked for small values of d .

In particular $C(2, 4) \geq 16$. Note that for polynomial automorphisms over a field, the maximal degree of an inverse can be obtained by a triangular map (Proposition 6.1.8 and Example 6.1.9), but that for an arbitrary ring this fails: by Proposition 6.4.1 the degree of a triangular invertible polynomial map with Jacobian determinant 1 in 2 variables of degree 4 is at most 13.

Because $C_\Delta(2, d) = d^2 - d + 1$, because the example above seems to obtain degree d^2 for the inverse of a polynomial map of degree d in 2 variables, and because $C(2, 3) = 9$, one could hope that $C(2, d) = d^2$. In fact, it isn't even true that $C(2, 4) = 16$. The following example shows that $C(2, 4) \geq 19$.

Example 6.5.3. Let F be the polynomial map

$$F := (X + \sum_{\substack{2 \leq i+j \leq 4 \\ i \neq 0, j \neq 4}} a_{ij} X^i Y^j + Y^4, Y + \sum_{\substack{2 \leq i+j \leq 4 \\ j \neq 4}} b_{ij} X^i Y^j)$$

on the polynomial ring R over \mathbb{Q} in all variables a_{ij}, b_{ij} . Take $d := \det(JF)$ and let I be the ideal generated by the coefficients of $d - 1$, considered as a polynomial in X and Y . Let \bar{R} be the \mathbb{Q} -algebra R/I . Then F induces a polynomial map \bar{F} over \bar{R} and $\det(J\bar{F}) = 1$.

One can now use a computer algebra system to compute a Gröbner basis of I and using it one can apply Proposition 1.5.5 to actually compute the inverse of F . It turns out that the degree of F^{-1} is 19. See Appendix A.

The example above is almost the “generic” example in 2 variables of degree 4. The problem is, however, that the Gröbner basis of the generic example could not be computed because it is too large. The restriction of taking some coefficients in the polynomial map fixed (either 0 or 1) is just enough to actually be able to compute this Gröbner basis.

Appendix A

Computations

This appendix contains two computations done with the computer algebra system MAPLE. This first one is the computation of the inverse of the generic polynomial map in two variables of degree three. See also Example 6.1.3. The second one is the computation of Example 6.5.3, which is “almost” the generic polynomial map of degree four in two variables.

A.1 Degree three

Computation of the degree of the inverse of a polynomial map in 2 variables of degree 3 with Jacobian determinant equal to 1.

Read some useful routines.

```
> read jacobi;
```

Create a generic polynomial map F of degree 3 in 2 variables.

```
> F := genmap(2, 3, [a, b], [x, y], 'v');
```

$$F := [a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 x y + a_6 y^2 + a_7 x^3 + a_8 x^2 y + a_9 x y^2 + a_{10} y^3, b_1 + b_2 x + b_3 y + b_4 x^2 + b_5 x y + b_6 y^2 + b_7 x^3 + b_8 x^2 y + b_9 x y^2 + b_{10} y^3]$$

Fix F in such a way that it is of the form $(x + \dots, y + \dots)$.

```
> F := subs(a[1]=0, a[2]=1, a[3]=0,
```

```
> b[1]=0, b[2]=0, b[3]=1, F);
```

$$F := [x + a_4 x^2 + a_5 x y + a_6 y^2 + a_7 x^3 + a_8 x^2 y + a_9 x y^2 + a_{10} y^3, y + b_4 x^2 + b_5 x y + b_6 y^2 + b_7 x^3 + b_8 x^2 y + b_9 x y^2 + b_{10} y^3]$$

Compute its Jacobian determinant.

```
> j := jacobdet(F, [x, y]);
```

$$\begin{aligned} j := & 1 + 3a_7x^4b_8 + b_8x^2 + 2a_9y^3b_6 - 3a_8x^4b_7 + 2b_9xy \\ & + 3a_5y^3b_{10} - 3a_5x^3b_7 + a_9y^2 - 2a_6y^2b_5 - 2a_8x^3b_4 \\ & + 2a_5y^2b_6 + b_5x + 2a_8xy + 2a_4x^3b_8 - 3a_{10}y^3b_5 \\ & - 2a_6y^3b_9 + 3a_7x^3b_5 + 2a_4x^2b_5 - 2a_5x^2b_4 + a_5y \\ & + 3a_9y^4b_{10} - 3a_{10}y^4b_9 + 2a_4x + 3b_{10}y^2 + 4a_4xb_6y \\ & + 4a_4x^2b_9y + 6a_4xb_{10}y^2 - a_5yb_8x^2 + a_5y^2b_9x \\ & + 6a_7x^2b_6y + 6a_7x^3b_9y + 9a_7x^2b_{10}y^2 + a_8x^2yb_5 \\ & + 4a_8xy^2b_6 + 3a_8x^2y^2b_9 + 6a_8xy^3b_{10} - a_9y^2b_5x \\ & - 3a_9y^2b_8x^2 - 4a_6yb_4x - 6a_6yb_7x^2 - 4a_6y^2b_8x \\ & - 4a_9x^2yb_4 - 6a_9x^3yb_7 - 6a_{10}y^2b_4x - 9a_{10}y^2b_7x^2 \\ & - 6a_{10}y^3b_8x + 3a_7x^2 + 2b_6y \end{aligned}$$

The ideal generated by the coefficients of j-1.

```
> J := [coeffs(j-1, [x, y])];
```

$$\begin{aligned} J := & [2a_4 + b_5, a_5 + 2b_6, -6a_{10}b_8 + 6a_8b_{10}, \\ & 2b_9 + 2a_8 + 4a_4b_6 - 4a_6b_4, \\ & -a_5b_8 + 6a_7b_6 - 6a_6b_7 - 4a_9b_4 + 4a_4b_9 + a_8b_5, \\ & -6a_{10}b_4 - 4a_6b_8 + a_5b_9 + 4a_8b_6 - a_9b_5 + 6a_4b_{10}, \\ & b_8 + 2a_4b_5 + 3a_7 - 2a_5b_4, 2a_5b_6 + a_9 - 2a_6b_5 + 3b_{10}, \\ & -2a_8b_4 + 3a_7b_5 + 2a_4b_8 - 3a_5b_7, \\ & -3a_{10}b_5 + 2a_9b_6 - 2a_6b_9 + 3a_5b_{10}, -3a_8b_7 + 3a_7b_8, \\ & -3a_{10}b_9 + 3a_9b_{10}, 6a_7b_9 - 6a_9b_7, \\ & 9a_7b_{10} - 9a_{10}b_7 - 3a_9b_8 + 3a_8b_9] \end{aligned}$$

It is enough to compute the degree of the inverse of F modulo the ideal J. In order to do this, a Grobner basis of I is needed.

```
> Tv := tdeg(op(v));
```

```
Tv := tdeg(a1, a2, a3, a4, a5, a6, a7, a8, a9, a10, b1, b2, b3, b4, b5, b6, b7, b8,
b9, b10)
```

```
> GJ := gbasis(J, Tv):
```

Now compute the formal inverse of F by computing successive powers of the derivation $X d/dF1 + Y d/dF2$ and substituting $x := 0$ and $y := 0$. The parameters 'GJ' and 'Tv' are needed in the next call because the computation has to be done modulo the ideal J; the parameter 'infinity' says that the computation should not terminate

before the full inverse has been computed; the parameter ‘true’ says that it is not needed to recompute det JF: this is guaranteed to be 1.

```
> G := formalinverse_mod(F, [x, y], GJ, Tv, infinity,
> true):
```

Now what is the leading form of G?

```
> map(wleadingform, G, [x, y]);
```

$$\begin{aligned}
& \left[\left(\frac{1179}{64} b_{10} a_{10}^2 a_7 + \frac{1179}{64} a_{10} a_8 b_{10}^2 + \frac{1179}{64} b_{10}^4 + \frac{1179}{64} a_9 b_{10}^3 \right) y^8 x + \left(-\frac{917}{16} b_9 b_{10}^2 b_8 + \frac{2751}{8} b_9 b_{10}^2 a_7 - \frac{917}{8} b_8 b_{10}^2 a_8 + \frac{8253}{16} b_7 b_{10}^3 \right) y^5 x^4 + \right. \\
& \left(\frac{917}{4} b_{10}^3 b_8 - \frac{917}{12} b_{10}^2 b_9^2 - \frac{917}{24} b_{10}^2 a_8 b_9 + \frac{2751}{8} b_{10}^3 a_7 \right) y^6 x^3 \\
& + \left(-\frac{917}{144} b_{10} b_9^2 b_8 + \frac{917}{24} b_8^2 b_{10}^2 - \frac{917}{16} b_9 b_7 b_{10}^2 \right) y^4 x^5 + \left(-\frac{393}{4} b_{10}^3 a_8 - \frac{1179}{8} a_{10} a_7 b_{10}^2 + \frac{131}{4} b_9 b_{10}^2 a_9 + \frac{131}{8} a_8 b_{10}^2 a_9 \right) y^7 x^2 + \\
& \left(\frac{917}{216} b_9 b_8^2 b_{10} + \frac{917}{24} b_7 b_8 b_{10}^2 - \frac{917}{36} b_{10} b_7 b_9^2 \right) y^3 x^6 \\
& + \left(-\frac{131}{32} b_7^2 b_9 b_{10} - \frac{131}{288} b_7 b_9^2 b_8 + \frac{131}{48} b_7 b_{10} b_8^2 \right) y x^8 \\
& + \left(-\frac{131}{8} b_9 b_7 b_8 b_{10} + \frac{393}{8} b_7^2 b_{10}^2 + \frac{131}{36} b_{10} b_8^3 \right) y^2 x^7 \\
& + \left(-\frac{131}{432} b_7^2 b_9^2 + \frac{131}{288} b_7^2 b_8 b_{10} + \frac{131}{2592} b_7 b_8^2 b_9 \right) x^9 + \\
& \left(-\frac{131}{64} a_{10}^2 a_8 b_{10} - \frac{131}{64} a_{10} b_{10}^3 - \frac{131}{64} a_{10} a_9 b_{10}^2 - \frac{131}{64} a_{10}^3 a_7 \right) y^9,
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{917}{24} b_{10} b_7 b_9^2 - \frac{917}{16} b_7 b_8 b_{10}^2 - \frac{917}{144} b_9 b_8^2 b_{10} \right) y^4 x^5 + \\
& \left(-\frac{393}{4} b_{10}^3 b_8 - \frac{1179}{8} b_{10}^3 a_7 + \frac{131}{8} b_{10}^2 a_8 b_9 + \frac{131}{4} b_{10}^2 b_9^2 \right) y^7 \\
& x^2 + \left(\frac{131}{8} b_7^2 b_9 b_{10} + \frac{131}{72} b_7 b_9^2 b_8 - \frac{131}{12} b_7 b_{10} b_8^2 \right) y^2 x^7 + \\
& \left(\frac{393}{16} b_{10}^3 a_8 + \frac{1179}{32} a_{10} a_7 b_{10}^2 - \frac{131}{32} a_8 b_{10}^2 a_9 - \frac{131}{16} b_9 b_{10}^2 a_9 \right) \\
& y^8 x + \left(-\frac{917}{24} b_8^2 b_{10}^2 + \frac{917}{144} b_{10} b_9^2 b_8 + \frac{917}{16} b_9 b_7 b_{10}^2 \right) y^5 x^4 + \left(\right. \\
& -\frac{2751}{8} b_7 b_{10}^3 + \frac{917}{12} b_8 b_{10}^2 a_8 + \frac{917}{24} b_9 b_{10}^2 b_8 - \frac{917}{4} b_9 b_{10}^2 a_7 \left. \right) \\
& y^6 x^3 + \left(\frac{131}{48} b_7^2 b_9^2 - \frac{131}{288} b_7 b_8^2 b_9 - \frac{131}{32} b_7^2 b_8 b_{10} \right) y x^8 \\
& + \left(-\frac{917}{8} b_7^2 b_{10}^2 + \frac{917}{24} b_9 b_7 b_8 b_{10} - \frac{917}{108} b_{10} b_8^3 \right) y^3 x^6 + \\
& \left(-\frac{131}{64} b_{10}^4 - \frac{131}{64} a_9 b_{10}^3 - \frac{131}{64} a_{10} a_8 b_{10}^2 - \frac{131}{64} b_{10} a_{10}^2 a_7 \right) y^9 \\
& + \left(-\frac{131}{96} b_7^3 b_{10} - \frac{131}{1296} b_7 b_8^3 + \frac{131}{288} b_7^2 b_9 b_8 \right) x^9]
\end{aligned}$$

A.2 Degree four

An example of an invertible polynomial map in 2 variables of degree 4 with Jacobian determinant equal to 1 whose inverse has degree 19.

Read some useful routines.

```
> read jacobi;
> d := 4;
```

$d := 4$

Create a generic polynomial map of degree 4 in 2 variables.

```
> F1 := ggenmap(2, d, [a, b], [x, y]);
```

$$\begin{aligned}
F1 := & [a_{0,0} + a_{1,0}x + a_{0,1}y + a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 + a_{3,0}x^3 \\
& + a_{2,1}x^2y + a_{1,2}xy^2 + a_{0,3}y^3 + a_{4,0}x^4 + a_{3,1}x^3y + a_{2,2}x^2y^2 \\
& + a_{1,3}xy^3 + a_{0,4}y^4, b_{0,0} + b_{1,0}x + b_{0,1}y + b_{2,0}x^2 + b_{1,1}xy \\
& + b_{0,2}y^2 + b_{3,0}x^3 + b_{2,1}x^2y + b_{1,2}xy^2 + b_{0,3}y^3 + b_{4,0}x^4 \\
& + b_{3,1}x^3y + b_{2,2}x^2y^2 + b_{1,3}xy^3 + b_{0,4}y^4]
\end{aligned}$$

Fix the map in such a way that it is of the form $(x + \dots, y + \dots)$ and choose some specific coefficients: the whole y -part of the first component is y^4 and the coefficient of y^4 in the second component is taken to be 0.

```
> substs := [seq(a[0,i]=0,i=0..d-1), a[0,d]=1,
> a[1,0]=1, b[0,0]=0,b[1,0]=0, b[0,1]=1, b[0,d]=0];
```

```
substs := [a0,0 = 0, a0,1 = 0, a0,2 = 0, a0,3 = 0, a0,4 = 1, a1,0 = 1,
b0,0 = 0, b1,0 = 0, b0,1 = 1, b0,4 = 0]
```

```
> F2 := subs(substs,F1);
```

$$F2 := [x + a_{2,0}x^2 + a_{1,1}xy + a_{3,0}x^3 + a_{2,1}x^2y + a_{1,2}xy^2 + a_{4,0}x^4 \\ + a_{3,1}x^3y + a_{2,2}x^2y^2 + a_{1,3}xy^3 + y^4, y + b_{2,0}x^2 + b_{1,1}xy \\ + b_{0,2}y^2 + b_{3,0}x^3 + b_{2,1}x^2y + b_{1,2}xy^2 + b_{0,3}y^3 + b_{4,0}x^4 \\ + b_{3,1}x^3y + b_{2,2}x^2y^2 + b_{1,3}xy^3]$$

The ideal generated by the coefficients of $\det JF2 - 1$.

```
> J2 := jacideal(F2,[x,y]);
```

$$\begin{aligned}
J2 := & [b_{1,1} + 2a_{2,0}, a_{1,1} + 2b_{0,2}, -4a_{3,1}b_{4,0} + 4a_{4,0}b_{3,1}, \\
& -3a_{3,1}b_{3,0} - 4a_{2,1}b_{4,0} + 4a_{4,0}b_{2,1} + 3a_{3,0}b_{3,1}, \\
& 3a_{1,3}b_{0,3} - 4b_{1,2}, -8b_{2,2}, a_{1,2} + 3b_{0,3} + 2a_{1,1}b_{0,2}, \\
& -8a_{2,2}b_{4,0} + 8a_{4,0}b_{2,2}, \\
& 4a_{3,1}b_{2,2} - 12a_{1,3}b_{4,0} - 4a_{2,2}b_{3,1} + 12a_{4,0}b_{1,3}, \\
& -2a_{1,2}b_{2,2} + 9a_{3,1}b_{0,3} + 2a_{2,2}b_{1,2} + 5a_{2,1}b_{1,3} \\
& - 5a_{1,3}b_{2,1} - 12b_{3,0}, \\
& -8b_{2,1} + a_{1,2}b_{1,3} + 6a_{2,2}b_{0,3} - a_{1,3}b_{1,2}, \\
& 8a_{3,1}b_{1,3} - 8a_{1,3}b_{3,1} - 16b_{4,0}a_{3,1}b_{2,1} + 6a_{3,0}b_{2,2} \\
& - a_{2,1}b_{3,1} - 6a_{2,2}b_{3,0} + 8a_{4,0}b_{1,2} - 8a_{1,2}b_{4,0}, -4b_{1,3}, \\
& 9a_{3,0}b_{1,3} - 5a_{1,2}b_{3,1} - 2a_{2,2}b_{2,1} + 12a_{4,0}b_{0,3} \\
& + 2a_{2,1}b_{2,2} + 5a_{3,1}b_{1,2} - 9a_{1,3}b_{3,0}, \\
& 2a_{2,1} + 4a_{2,0}b_{0,2} + 2b_{1,2}, \\
& -8b_{2,0} + 4a_{2,2}b_{0,2} - 2a_{1,3}b_{1,1} + 6a_{2,1}b_{0,3} + 2a_{1,1}b_{1,3}, \\
& -4a_{1,3}b_{2,2} + 4a_{2,2}b_{1,3} - 12b_{3,1}, -3a_{1,1}b_{3,0} + 2a_{2,0}b_{2,1} \\
& + 3a_{3,0}b_{1,1} - 2a_{2,1}b_{2,0} + b_{3,1} + 4a_{4,0}, \\
& 3a_{3,0} - 2a_{1,1}b_{2,0} + b_{2,1} + 2a_{2,0}b_{1,1}, a_{2,1}b_{1,1} + 4a_{2,0}b_{1,2} \\
& + 6a_{3,0}b_{0,2} - a_{1,1}b_{2,1} - 4a_{1,2}b_{2,0} + 2b_{2,2} + 3a_{3,1}, \\
& 3a_{2,1}b_{1,2} - 6a_{1,3}b_{2,0} - 3a_{1,2}b_{2,1} + 6a_{2,0}b_{1,3} + 6a_{3,1}b_{0,2} \\
& + 9a_{3,0}b_{0,3}, a_{1,3} + 2a_{1,2}b_{0,2} + 3a_{1,1}b_{0,3}, 3a_{3,0}b_{2,1} \\
& - 4a_{1,1}b_{4,0} + 4a_{4,0}b_{1,1} - 2a_{3,1}b_{2,0} - 3a_{2,1}b_{3,0} \\
& + 2a_{2,0}b_{3,1}, 6a_{3,0}b_{1,2} + 2a_{3,1}b_{1,1} - 6a_{1,2}b_{3,0} \\
& - 2a_{1,1}b_{3,1} + 4a_{2,0}b_{2,2} - 4a_{2,2}b_{2,0} + 8a_{4,0}b_{0,2}, \\
& 3a_{1,2}b_{0,3} + 2a_{1,3}b_{0,2} - 4b_{1,1}, \\
& 2a_{2,2} + a_{1,1}b_{1,2} + 6a_{2,0}b_{0,3} - a_{1,2}b_{1,1} + 3b_{1,3} + 4a_{2,1}b_{0,2} \\
&]
\end{aligned}$$

Some of the above relations are of the form ‘variable - polynomial in other variables’. In order to make the Grobner basis computation easier, I’ll solve these first.

```

> EQ := eqnl(J2, indets(J2), 's');
> while EQ <> [] do
>   J2 := subs(s[1], JI2);
>   substs := [op(substs), s[1]];
>   EQ := eqnl(J2, indets(J2), 's');
> od:

```

EQ now contains all easy substitutions. The next line applies them to the polynomial map.

```
> F3 := expand(repsubs(substs,F2));
```

$$\begin{aligned}
F3 := & [16x^3yb_{0,2}^9 - 26x^4b_{0,2}^{10}a_{1,2} - \frac{1}{16}x^4a_{1,2}^5b_{0,2}^2 \\
& - 16x^3yb_{0,2}^7a_{1,2} + \frac{5}{8}x^4a_{1,2}^4b_{0,2}^4 - \frac{11}{8}x^4b_{0,2}^6a_{1,2}^3 \\
& + \frac{15}{2}x^4b_{0,2}^8a_{1,2}^2 + 24x^4b_{0,2}^{12} - \frac{11}{48}x^3ya_{1,2}^4b_{0,2} \\
& - \frac{11}{6}x^3yb_{0,2}^3a_{1,2}^3 + 9x^3yb_{0,2}^5a_{1,2}^2 - 2x^3a_{1,2}b_{0,2}^6 \\
& + 5x^2yb_{0,2}^3a_{1,2} - \frac{5}{4}x^2ya_{1,2}^2b_{0,2} + \frac{1}{8}x^2a_{1,2}^2 - 2x^2b_{0,2}^4 \\
& - 4xy^3a_{1,2}b_{0,2} - 18x^2y^2b_{0,2}^4a_{1,2} + 3x^2y^2a_{1,2}^2b_{0,2}^2 \\
& + \frac{1}{2}x^2a_{1,2}b_{0,2}^2 + \frac{1}{48}x^3a_{1,2}^4 + \frac{1}{6}x^3a_{1,2}^3b_{0,2}^2 \\
& + \frac{1}{2}x^3b_{0,2}^4a_{1,2}^2 - 4x^2yb_{0,2}^5 + x + 24x^2y^2b_{0,2}^6 - 2b_{0,2}xy \\
& + \frac{1}{256}x^4a_{1,2}^6 + y^4 + a_{1,2}xy^2 + 8xy^3b_{0,2}^3, \frac{7}{2}x^2yb_{0,2}^4a_{1,2}^2 \\
& - \frac{1}{4}x^2ya_{1,2}^3b_{0,2}^2 - 14x^2ya_{1,2}b_{0,2}^6 - 6xy^2b_{0,2}^3a_{1,2} \\
& + xy^2a_{1,2}^2b_{0,2} - \frac{16}{3}x^3b_{0,2}^{11} + 16x^2yb_{0,2}^8 - \frac{1}{3}y^3a_{1,2} \\
& + b_{0,2}y^2 - xy a_{1,2}b_{0,2}^2 - \frac{41}{12}x^3a_{1,2}^3b_{0,2}^5 + \frac{5}{16}x^3a_{1,2}^4b_{0,2}^3 \\
& + \frac{11}{192}x^3a_{1,2}^5b_{0,2} + \frac{4}{3}y^3b_{0,2}^2 + 4xyb_{0,2}^4 - \frac{1}{4}xy a_{1,2}^2 \\
& + \frac{1}{16}x^2a_{1,2}^3b_{0,2} - \frac{3}{2}x^2a_{1,2}^2b_{0,2}^3 + 3x^2b_{0,2}^5a_{1,2} + y \\
& + \frac{16}{3}x^3b_{0,2}^9a_{1,2} + 8xy^2b_{0,2}^5 + 4x^3b_{0,2}^7a_{1,2}^2]
\end{aligned}$$

This map is the map of degree 4 that is going to have a large inverse. Its Jacobian ideal has in fact already been computed (J2), but the substitutions in EQ have to be applied to it. Actually, it's easier to recompute the ideal of the coefficients of JF3 - 1.

```
> J3 := jacideal(F3,[x,y]);
```

$$\begin{aligned}
J3 := & \left[-\frac{3}{2} a_{1,2}^5 b_{0,2}^3 - \frac{695}{3} b_{0,2}^7 a_{1,2}^3 + 320 b_{0,2}^{13} + \frac{130}{3} a_{1,2}^4 b_{0,2}^5 \right. \\
& - \frac{7}{96} a_{1,2}^6 b_{0,2} - 720 b_{0,2}^{11} a_{1,2} + 580 b_{0,2}^9 a_{1,2}^2, \\
& 120 a_{1,2}^2 b_{0,2}^6 - \frac{25}{8} a_{1,2}^4 b_{0,2}^2 - \frac{1}{16} a_{1,2}^5 - 120 b_{0,2}^8 a_{1,2} \\
& - \frac{45}{2} a_{1,2}^3 b_{0,2}^4, -\frac{130}{3} b_{0,2}^{11} a_{1,2}^3 - \frac{319}{192} a_{1,2}^7 b_{0,2}^3 \\
& - 4640 b_{0,2}^{15} a_{1,2} + \frac{877}{24} a_{1,2}^5 b_{0,2}^7 + \frac{1}{32} a_{1,2}^8 b_{0,2} \\
& + 2440 b_{0,2}^{13} a_{1,2}^2 + \frac{271}{32} a_{1,2}^6 b_{0,2}^5 - \frac{525}{2} b_{0,2}^9 a_{1,2}^4 \\
& + 2560 b_{0,2}^{17}, -\frac{415}{6} a_{1,2}^4 b_{0,2}^6 + 280 b_{0,2}^{10} a_{1,2}^2 \\
& + \frac{59}{48} a_{1,2}^6 b_{0,2}^2 - 960 b_{0,2}^{12} a_{1,2} + \frac{410}{3} b_{0,2}^8 a_{1,2}^3 + 640 b_{0,2}^{14} \\
& - \frac{1}{64} a_{1,2}^7 + \frac{17}{6} a_{1,2}^5 b_{0,2}^4, -\frac{187}{48} a_{1,2}^7 b_{0,2}^6 - 3520 b_{0,2}^{18} a_{1,2} \\
& - \frac{377}{48} a_{1,2}^6 b_{0,2}^8 + \frac{497}{768} a_{1,2}^8 b_{0,2}^4 + 2480 b_{0,2}^{16} a_{1,2}^2 \\
& + \frac{773}{12} a_{1,2}^5 b_{0,2}^{10} + \frac{109}{3072} a_{1,2}^9 b_{0,2}^2 - \frac{2140}{3} b_{0,2}^{14} a_{1,2}^3 \\
& - \frac{40}{3} b_{0,2}^{12} a_{1,2}^4 + 1792 b_{0,2}^{20}, -\frac{35}{96} a_{1,2}^6 b_{0,2}^4 \\
& - 245 b_{0,2}^{10} a_{1,2}^3 + \frac{211}{768} a_{1,2}^7 b_{0,2}^2 - 560 b_{0,2}^{14} a_{1,2} \\
& + \frac{475}{4} b_{0,2}^8 a_{1,2}^4 + 400 b_{0,2}^{12} a_{1,2}^2 - \frac{379}{16} a_{1,2}^5 b_{0,2}^6 \\
& \left. + 320 b_{0,2}^{16} - \frac{1}{256} a_{1,2}^8 \right] \\
> \text{ T3} := \text{tdeg}(\text{op}(\text{indets}(J3))) ; \\
& \text{ T3} := \text{tdeg}(a_{1,2}, b_{0,2})
\end{aligned}$$

So only two variables have survived! Computations should be done modulo J3, so compute a Grobner basis of this ideal.

```
> G3 := gbasis(J3, T3);
```

$$\begin{aligned}
G3 := & [a_{1,2}^8, -1920 a_{1,2}^2 b_{0,2}^6 + 50 a_{1,2}^4 b_{0,2}^2 + a_{1,2}^5 \\
& + 1920 b_{0,2}^8 a_{1,2} + 360 a_{1,2}^3 b_{0,2}^4, a_{1,2}^7 b_{0,2}^2, \\
& 1600 a_{1,2}^4 b_{0,2}^6 - 123 a_{1,2}^6 b_{0,2}^2 - 7 a_{1,2}^7 - 444 a_{1,2}^5 b_{0,2}^4, \\
& 17852 b_{0,2}^5 a_{1,2}^5 - 7561 a_{1,2}^6 b_{0,2}^3 - 319 a_{1,2}^7 b_{0,2}, \\
& a_{1,2}^6 b_{0,2}^4, 15360 b_{0,2}^{13} - 11360 b_{0,2}^7 a_{1,2}^3 + 4240 a_{1,2}^4 b_{0,2}^5 \\
& + 121 a_{1,2}^5 b_{0,2}^3]
\end{aligned}$$

Let's reduce the (coefficients of the) map itself modulo G3. In that way, one sees that the degree of the map is really 4.

```

> F := collect(map(normalf,F3,G3,T3),[x,y],
> distributed);

```


$$\begin{aligned}
F := & [(-2b_{0,2}^4 + \frac{1}{2}b_{0,2}^2a_{1,2} + \frac{1}{8}a_{1,2}^2)x^2 \\
& + (-18b_{0,2}^4a_{1,2} + 3a_{1,2}^2b_{0,2}^2 + 24b_{0,2}^6)y^2x^2 \\
& + (-4a_{1,2}b_{0,2} + 8b_{0,2}^3)y^3x \\
& + (-\frac{5}{4}b_{0,2}^5a_{1,2}^2 - 4b_{0,2}^5 + 5a_{1,2}b_{0,2}^3)x^2y + (16b_{0,2}^9 \\
& + 9b_{0,2}^5a_{1,2}^2 - \frac{11}{48}a_{1,2}^4b_{0,2} - 16b_{0,2}^7a_{1,2} - \frac{11}{6}b_{0,2}^3a_{1,2}^3) \\
& yx^3 + (24b_{0,2}^{12} + \frac{277}{640}a_{1,2}^5b_{0,2}^2 - 15b_{0,2}^6a_{1,2}^3 \\
& + \frac{229}{48}a_{1,2}^4b_{0,2}^4 + \frac{13}{960}a_{1,2}^6)x^4 \\
& + (\frac{1}{48}a_{1,2}^4 + \frac{1}{6}b_{0,2}^2a_{1,2}^3 + \frac{1}{2}a_{1,2}^2b_{0,2}^4 - 2b_{0,2}^6a_{1,2})x^3 \\
& - 2b_{0,2}xy + a_{1,2}xy^2 + x + y^4, \\
& (3a_{1,2}b_{0,2}^5 - \frac{3}{2}b_{0,2}^3a_{1,2}^2 + \frac{1}{16}b_{0,2}a_{1,2}^3)x^2 + b_{0,2}y^2 \\
& + (-14b_{0,2}^6a_{1,2} - \frac{1}{4}b_{0,2}^2a_{1,2}^3 + 16b_{0,2}^8 + \frac{7}{2}a_{1,2}^2b_{0,2}^4)x^2y \\
& + (-\frac{53}{12}a_{1,2}^3b_{0,2}^5 + \frac{28}{3}b_{0,2}^7a_{1,2}^2 + \frac{157}{2880}a_{1,2}^5b_{0,2} - \frac{16}{3}b_{0,2}^{11} \\
& + \frac{25}{144}a_{1,2}^4b_{0,2}^3)x^3 + (-\frac{1}{4}a_{1,2}^2 + 4b_{0,2}^4 - b_{0,2}^2a_{1,2})xy \\
& + (8b_{0,2}^5 + b_{0,2}a_{1,2}^2 - 6a_{1,2}b_{0,2}^3)xy^2 + (\frac{4}{3}b_{0,2}^2 - \frac{1}{3}a_{1,2})y^3 \\
& + y]
\end{aligned}$$

What is the degree of F?

```
> map(wdegree, F, [x, y]);
[4, 3]
```

The first component has degree 4, the second one degree 3. Now start the computation of the inverse. Don't show it, because it is probably very large.

```
> H := formalinverse_mod(F, [x, y], G3, T3, infinity,
> true);
```

The degree of (both components of) the inverse.

```
> map(wdegree, H, [x, y]);
[19, 16]
```

The degree of the first component of the inverse is 19, of the second component 16. Its leading form looks like this:

```
> map(wleadingform, H, [x, y]);
```

$$\left[\left(\frac{3893872974671}{112441893120} a_{1,2}^6 b_{0,2}^3 - \frac{150950066851}{224883786240} a_{1,2}^7 b_{0,2} \right) y^{19}, \right. \\ \left. \left(\frac{17753243797}{59969009664} a_{1,2}^7 b_{0,2} - \frac{1290836398589}{119938019328} a_{1,2}^6 b_{0,2}^3 \right) y^{16} \right]$$

Curriculum Vitae

The author was born on April 2nd, 1971 in Macharen. He started studying computer science in 1989 at the University of Nijmegen and graduated cum laude, writing a master's thesis on intersection type disciplines under the supervision of dr. S. van Bakel in 1993. He continued this research during a half year stay in Turin, under the supervision of prof.dr. M. Dezani-Ciangiagini. After returning to Nijmegen in 1994, the author commenced studying mathematics, writing a master's thesis on cyclotomic number fields under the supervision of prof.dr. F. Keune in 1996. After his graduation cum laude in mathematics, he was a Ph.D. student at Nijmegen, which culminated in this thesis, written under the supervision of dr. A. van den Essen. The author currently works as a visiting assistant professor at New Mexico State University in Las Cruces.

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- Weitzenböck derivation, *see* derivation